Phys 543 Fall 2017

Homework 4

Solutions

$\boxed{4.1}$ a) Relevant states: $|j=\frac{1}{2}, m_j=\pm\frac{1}{2}\rangle$, $|j=\frac{3}{2}, m_j=\pm\frac{1}{2}, \pm\frac{3}{2}\rangle$

$$A\vec{L}\cdot\vec{S} = \frac{A}{2}\left((\vec{L}+\vec{S})^2 - L^2 - S^2\right) = \frac{A\hbar^2}{2}\left[j(j+1) - \ell(\ell+1) - s(s+1)\right]$$

$$= \frac{A\hbar^2}{2}\begin{cases} -2, & j=1/2 \\ +1, & j=3/2 \end{cases}$$

Matrix: $$\frac{A\hbar^2}{2}\begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && -2 &&& \\ &&& 1 && \\ &&&& -2 & \\ &&&&& 1 \end{pmatrix}$$

$$\gamma\vec{\mu}\cdot\vec{B} = \frac{e}{2m}(\vec{L}+2\vec{S})\cdot\vec{B} = \frac{e}{2m}(L_z + 2S_z)B = \mu_B B(m_\ell + 2m_s)$$

$$= \frac{eB\hbar}{2m}\Big[2|1\,\tfrac{1}{2}\rangle\langle 1\,\tfrac{1}{2}| + 0\,|1\,-\tfrac{1}{2}\rangle\langle 1\,-\tfrac{1}{2}| + 1\,|0\,\tfrac{1}{2}\rangle\langle 0\,\tfrac{1}{2}| + (-1)|0\,-\tfrac{1}{2}\rangle\langle 0\,-\tfrac{1}{2}| + 0\,|-1\,\tfrac{1}{2}\rangle\langle -1\,\tfrac{1}{2}| + (-2)|-1\,-\tfrac{1}{2}\rangle\langle -1\,-\tfrac{1}{2}|\Big]$$ (in $|m_\ell, m_s\rangle$ basis)

Now use Clebsch-Gordon coefficients to write

$$|1\,\tfrac{1}{2}\rangle = |j=3/2, m_j=3/2\rangle$$

$$|0\,\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|j=3/2, m_j=1/2\rangle - \sqrt{\tfrac{1}{3}}\,|j=1/2, m_j=1/2\rangle$$

$$|0\,-\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|j=3/2; m_j=-1/2\rangle + \sqrt{\tfrac{1}{3}}\,|j=1/2, m_j=-1/2\rangle$$

$$|-1\,-\tfrac{1}{2}\rangle = |j=3/2, m_j=-3/2\rangle$$

$$\Rightarrow \gamma\vec{\mu}\cdot\vec{B} = \mu_B B\Big[2|\tfrac{3}{2}\,\tfrac{3}{2}\rangle\langle\tfrac{3}{2}\,\tfrac{3}{2}| + 1\left(\sqrt{\tfrac{2}{3}}|\tfrac{3}{2}\,\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right)\left(\sqrt{\tfrac{2}{3}}\langle\tfrac{3}{2}\,\tfrac{1}{2}| - \sqrt{\tfrac{1}{3}}\langle\tfrac{1}{2}\,\tfrac{1}{2}|\right) + (-1)\left(\sqrt{\tfrac{2}{3}}|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\right)\left(\sqrt{\tfrac{2}{3}}\langle\tfrac{3}{2}\,-\tfrac{1}{2}| + \sqrt{\tfrac{1}{3}}\langle\tfrac{1}{2}\,-\tfrac{1}{2}|\right) + (-2)|\tfrac{3}{2}\,-\tfrac{3}{2}\rangle\langle\tfrac{3}{2}\,-\tfrac{3}{2}|\Big]$$

(in $|j, m_j\rangle$ basis)

Matrix:

$$\frac{eB\hbar}{2m}\begin{pmatrix} 2 & & & & & \\ & 2/3 & -\sqrt{2}/3 & & & \\ & -\sqrt{2}/3 & 1/3 & & & \\ & & & -2/3 & -\sqrt{2}/3 & \\ & & & -\sqrt{2}/3 & -1/3 & \\ & & & & & -2 \end{pmatrix}$$

Full matrix:

Block 1, Block 2, Block 3, Block 4

$$\hat{H} = \begin{pmatrix} \alpha+2\beta & & & & & \\ & \alpha+\frac{2}{3}\beta & -\frac{\sqrt{2}}{3}\beta & & & \\ & -\frac{\sqrt{2}}{3}\beta & -2\alpha+\frac{1}{3}\beta & & & \\ & & & \alpha-\frac{2}{3}\beta & -\frac{\sqrt{2}}{3}\beta & \\ & & & -\frac{\sqrt{2}}{3}\beta & -2\alpha-\frac{1}{3}\beta & \\ & & & & & \alpha-2\beta \end{pmatrix}$$

b) Eigenvalue of Block 1:

$$E_1 = \alpha + 2\beta$$

Eigenvalues of Block 2:

$$E_2 = \frac{1}{2}\left(-\alpha+\beta+\sqrt{9\alpha^2+2\alpha\beta+\beta^2}\right)$$

$$E_3 = \frac{1}{2}\left(-\alpha+\beta-\sqrt{9\alpha^2+2\alpha\beta+\beta^2}\right)$$

Eigenvalues of Block 3:

$$E_4 = \frac{1}{2}\left(-\alpha-\beta+\sqrt{9\alpha^2-2\alpha\beta+\beta^2}\right)$$

$$E_5 = \frac{1}{2}\left(-\alpha-\beta-\sqrt{9\alpha^2-2\alpha\beta+\beta^2}\right)$$

Eigenvalue of Block 4:

$$E_6 = \alpha - 2\beta$$

c) for $\alpha \gg \beta$: $\sqrt{9\alpha^2+2\alpha\beta+\beta^2} = 3\alpha\sqrt{1+\frac{2\alpha\beta+\beta^2}{9\alpha^2}}$

$$\approx 3\alpha\left(1+\frac{\alpha\beta}{9\alpha^2}\right) = 3\alpha + \beta/3$$

$$\sqrt{9\alpha^2-2\alpha\beta+\beta^2} \approx 3\alpha - \beta/3$$

For $\alpha \ll \beta$: $\sqrt{9\alpha^2+2\alpha\beta+\beta^2} = \beta\sqrt{1+\frac{9\alpha^2+2\alpha\beta}{\beta^2}}$

$$\approx \beta\left(1+\frac{\alpha\beta}{\beta^2}\right) = \beta+\alpha$$

$$\sqrt{9\alpha^2-2\alpha\beta+\beta^2} \approx \beta - \alpha$$

For $\alpha >> \beta$: $E_2 \approx \alpha + \frac{2}{3}\beta$

$E_3 \approx -2\alpha + \frac{1}{3}\beta$

$E_4 \approx \alpha - \frac{2}{3}\beta$

$E_5 \approx -2\alpha - \frac{1}{3}\beta$

for $\alpha << \beta$: $E_2 \approx \beta$

$E_3 \approx -\alpha$

$E_4 \approx -\alpha$

$E_5 \approx -\beta$

d) - For $\alpha >> \beta$, level $j = \frac{3}{2}$ $(E_{j=3/2} = +\alpha)$

splits into four: $E_1 = \alpha + 2\beta$ $(m_j = 3/2)$

$E_2 = \alpha + \frac{2}{3}\beta$ $(m_j = 1/2)$

$E_4 = \alpha - \frac{2}{3}\beta$ $(m_j = -1/2)$

$E_6 = \alpha - 2\beta$ $(m_j = -3/2)$

$\Rightarrow E_{j=\frac{3}{2}, m_j} = \frac{4}{3}\beta m_j = \frac{4}{3}\mu_B B m_j$

level $j = \frac{1}{2}$ $(E_{j=1/2} = -2\alpha)$

splits into two: $E_3 = -2\alpha + \frac{1}{3}\beta$ $(m_j = 1/2)$

$E_5 = -2\alpha - \frac{1}{3}\beta$ $(m_j = -1/2)$

$\Rightarrow E_{j=\frac{1}{2}, m_j} = \frac{2}{3}\beta m_j = \frac{2}{3}\mu_B B m_j$

Verify: $g_j = 1 + \frac{j(j+1) + s(s+1) - \ell(\ell+1)}{2j(j+1)} = \begin{cases} 4/3, & j = 3/2 \\ 2/3, & j = 1/2 \end{cases}$

e) For $\beta >> \alpha$, five asymptotic energy levels

$E_1 = \alpha + 2\beta$ $(m_\ell = 1, m_s = 1/2)$

$E_2 = \beta$ $(m_\ell = 0, m_s = 1/2)$

$E_3 = E_4 = -\alpha$ $(m_\ell = 1, m_s = -1/2$ and $m_\ell = -1, m_s = 1/2)$

$E_5 = -\beta$ $(m_\ell = 0, m_s = -1/2)$

$E_6 = \alpha - 2\beta$ $(m_\ell = -1, m_s = -1/2)$

$\boxed{4.2}$ a) Using the table of Clebsch-Gordon coefficients,

$$|\psi_{AB}(0)\rangle = |S_A = \tfrac{1}{2}, S_B = 1, s = \tfrac{1}{2}, m_s = \tfrac{1}{2}\rangle$$

$$= \sqrt{\tfrac{2}{3}}\, |m_A = -\tfrac{1}{2}, m_B = 1\rangle - \sqrt{\tfrac{1}{3}}\, |m_A = \tfrac{1}{2}, m_B = 0\rangle$$

b) $$\left.\begin{aligned} p_1 = \tfrac{2}{3}, \ |\psi_1(0)\rangle = |\downarrow\rangle \\ p_2 = \tfrac{1}{3}, \ |\psi_2(0)\rangle = |\uparrow\rangle \end{aligned}\right\} \Rightarrow \rho_A(0) = \tfrac{2}{3}|\downarrow\rangle\langle\downarrow| + \tfrac{1}{3}|\uparrow\rangle\langle\uparrow| = \begin{pmatrix} 1/3 & 0 \\ 0 & 2/3 \end{pmatrix}$$

c) $$\rho_{AB}(0) = \tfrac{2}{3}|\downarrow,1\rangle\langle\downarrow,1| - \tfrac{\sqrt{2}}{3}|\downarrow,1\rangle\langle\uparrow,0|$$
$$- \tfrac{\sqrt{2}}{3}|\uparrow,0\rangle\langle\downarrow,1| + \tfrac{1}{3}|\uparrow,0\rangle\langle\uparrow,0|$$

$$\rho_A(0) = \mathrm{Tr}_B[\rho_{AB}(0)] =$$

$$\langle m_B = 1|\rho_{AB}(0)|m_B = 1\rangle + \langle m_B = 0|\rho_{AB}(0)|m_B = 0\rangle + \langle m_B = -1|\rho_{AB}(0)|m_B = -1\rangle$$

$$= \tfrac{2}{3}|\downarrow\rangle\langle\downarrow| + \tfrac{1}{3}|\uparrow\rangle\langle\uparrow|$$

d) $$H = -\gamma\vec{S}\cdot\vec{B} = -\gamma B S_{Ax} = -\gamma B\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Evolution operator

$$e^{-\frac{i}{\hbar}Ht} = e^{i\gamma B\sigma_x t} = \hat{\mathbb{1}}\cos\gamma Bt + i\sigma_x \sin\gamma Bt = \begin{pmatrix} \cos\gamma Bt & i\sin\gamma Bt \\ i\sin\gamma Bt & \cos\gamma Bt \end{pmatrix}$$

$$|\psi_1(t)\rangle = U\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} i\sin\gamma Bt \\ \cos\gamma Bt \end{pmatrix}$$

$$|\psi_2(t) = U\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\gamma Bt \\ i\sin\gamma Bt \end{pmatrix}$$

$$\rho_A(t) = \tfrac{2}{3}|\psi_1(t)\rangle\langle\psi_1(t)| + \tfrac{1}{3}|\psi_2(t)\rangle\langle\psi_2(t)|$$
$$= \tfrac{1}{3}\begin{pmatrix} 2\sin^2\gamma Bt + \cos^2\gamma Bt & i\cos\gamma Bt\sin\gamma Bt \\ -i\cos\gamma Bt\sin\gamma Bt & 2\cos^2\gamma Bt + \sin^2\gamma Bt \end{pmatrix}$$

e) $$\rho_A(t) = e^{-\frac{i}{\hbar}Ht}\rho_A(0)\, e^{\frac{i}{\hbar}Ht} \qquad \text{same as in (d)}$$

f) $$\mathrm{Tr}\,\rho_A(t)^2 = \tfrac{1}{9}\Big[(2\sin^2\gamma Bt + \cos^2\gamma Bt)^2 + 2(\cos\gamma Bt\sin\gamma Bt)^2$$
$$+ (2\cos^2\gamma Bt + \sin^2\gamma Bt)^2\Big]$$
$$= \tfrac{1}{9}\Big[(1 + \sin^2\gamma Bt)^2 + 2\cos^2\gamma Bt\sin^2\gamma Bt + (1 + \cos^2\gamma Bt)^2\Big]$$
$$= \tfrac{1}{9}\Big[2 + 2\sin^2\gamma Bt + 2\cos^2\gamma Bt + (\sin^2\gamma Bt + \cos^2\gamma Bt)^2\Big] = \tfrac{5}{9}$$

g) Method 1: $|R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$; $|L\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$

$$pr(|R\rangle) = \frac{2}{3}|\langle R|\psi_1(t)\rangle|^2 + \frac{1}{3}|\langle R|\psi_2(t)\rangle|^2$$

$$= \frac{1}{3}(\sin\gamma Bt - \cos\gamma Bt)^2 + \frac{1}{6}(\cos\gamma Bt + \sin\gamma Bt)^2$$

$$= \frac{1}{2} - \frac{1}{3}\sin\gamma Bt\cos\gamma Bt = \frac{1}{2} - \frac{1}{6}\sin 2\gamma Bt$$

$$pr(|L\rangle) = \frac{2}{3}|\langle L|\psi_1(t)\rangle|^2 + \frac{1}{3}|\langle L|\psi_2(t)\rangle|^2$$

$$= \frac{1}{3}(\sin\gamma Bt + \cos\gamma Bt)^2 + \frac{1}{6}(\cos\gamma Bt - \sin\gamma Bt)^2$$

$$= \frac{1}{2} + \frac{1}{6}\sin 2\gamma Bt$$

Method 2:

$$pr(|R\rangle) = \langle R|\rho_A(t)|R\rangle$$

$$= \frac{1}{6}(1 \;\; -i)\begin{pmatrix}2\sin^2\gamma Bt + \cos^2\gamma Bt - \cos\gamma Bt\sin\gamma Bt\\ -i\cos\gamma Bt\sin\gamma Bt + 2i\cos^2\gamma Bt + i\sin^2\gamma Bt\end{pmatrix}$$

$$= \frac{1}{6}(3\sin^2\gamma Bt + 3\cos^2\gamma Bt - 2\cos\gamma Bt\sin\gamma Bt)$$

$$= \frac{1}{2} - \frac{1}{6}\sin 2\gamma Bt$$

$$pr(L) = \langle L|\rho_A(t)|L\rangle = \frac{1}{2} + \frac{1}{6}\sin\gamma Bt.$$