University of Calgary
Winter semester 2017

PHYS 543: Quantum Mechanics II

Homework assignment 4

Due November 15, 2017

Problem 4.1. Perform full calculation of the energy eigenvalues of an electron in an atom as a function of the applied magnetic field. In addition to the Zeeman interaction, the spin-orbit interaction is present, so the total Hamiltonian is

$$\hat{H} = A\hat{\vec{L}} \cdot \hat{\vec{S}} - \hat{\vec{\mu}} \cdot \vec{B}.$$

The quantization axis z is chosen along the magnetic field; $A > 0$ is constant. The orbital angular momentum $l = 1$. The gyromagnetic ratios for the orbital and spin angular momenta are, respectively,

$$\gamma_l = -\frac{e}{2M} = -\frac{\mu_B}{\hbar}; \quad \gamma_s = -\frac{e}{M} = -\frac{2\mu_B}{\hbar},$$

where μ_B is the Bohr magneton, and the negative sign accounts for the negative charge of the electron.

a) Find the matrix of the Hamiltonian in the $\{|j, m_j\rangle]\}$ basis, with (j, m_j) taking on all possible values for $l = 1, s = \frac{1}{2}$:

$$|v_1\rangle = \left|j = \frac{3}{2}, m_j = \frac{3}{2}\right\rangle;$$
$$|v_2\rangle = \left|j = \frac{3}{2}, m_j = \frac{1}{2}\right\rangle;$$
$$|v_3\rangle = \left|j = \frac{1}{2}, m_j = \frac{1}{2}\right\rangle;$$
$$|v_4\rangle = \left|j = \frac{3}{2}, m_j = -\frac{1}{2}\right\rangle;$$
$$|v_5\rangle = \left|j = \frac{1}{2}, m_j = -\frac{1}{2}\right\rangle;$$
$$|v_6\rangle = \left|j = \frac{3}{2}, m_j = -\frac{3}{2}\right\rangle.$$

Use the notation

$$\alpha = \frac{A\hbar^2}{2}; \ \beta = \mu_B B.$$

Hint: First write the first term of the Hamiltonian in the $\{|j, m_j\rangle]\}$ basis, and the second term in the $\{|m_l, m_s\rangle\}$ basis. Then, knowing that the two bases are related via the Clebsch-Gordan coefficients, rewrite the second term in the $\{|j, m_j\rangle]\}$ basis using the resolution of the identity.

b) Find the eigenvalues $E_i(\alpha, \beta)$ of the Hamiltonian.
Hint: Rather than solving the characteristic equation for the entire 6×6 matrix, notice that the matrix is block-diagonal and diagonalize individual blocks.

c) Decompose the expressions for the energy eigenvalues into a power series in β for $\beta \ll \alpha$ (weak Zeeman effect) and in α for $\beta \gg \alpha$ (strong Zeeman effect) up to the two leading terms.

d) In the regime of weak Zeeman effect, show that each of the levels with a certain value of j splits according to

$$E_{j,m_j}(B) \approx E_j(B=0) + \mu_B B g_j m_j,$$

where

$$g_j = 1 + \frac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1)}.$$

e) In the regime of strong Zeeman effect, show that energy levels asymptotically tend to

$$E_{m_l,m_s} \approx \mu_B B(m_l + 2m_s) + O(\alpha).$$

f) Use software to plot all six $E_i(\alpha, \beta)$ as a function of β in the range $[0, 5\alpha]$. Also plot the asymptotic behavior for the strong Zeeman effect case by dashed lines.

Problem 4.2. Alice has a spin-$\frac{1}{2}$ particle, Bob has a spin-1 particle. The initial ($t = 0$) state of the two particles is $|\Psi_{AB}(0)\rangle = \left|s = \frac{1}{2}, m_s = \frac{1}{2}\right\rangle$, where s and m_s are the quantum numbers corresponding to the observables $(\hat{\vec{S}}_A + \hat{\vec{S}}_B)^2$ and $\hat{S}_{Az} + \hat{S}_{Bz}$, respectively. Alice's particle, whose gyromagnetc ratio is γ, is subjected to a magnetic field $\vec{B}$ oriented along the x axis.

a) Find the state $|\Psi_{AB}(0)\rangle$ in the canonical basis. Tables of Clebsch-Gordan coefficients can be used.

b) Write the statistical ensemble $\{p_i, |\psi_i(0)\rangle\}$ describing the state of Alice's particle at $t = 0$ if Bob's particle is not accessible for measurement. Find the density operator $\hat{\rho}_A(0)$ of Alice's particle from that ensemble.

c) Find $\hat{\rho}_B(0)$ from $|\Psi_{AB}(0)\rangle$ using the partial trace formalism. Check that your result is consistent with part (b).
Hint: It is more convenient to work in the Dirac notation, rather than the matrix notation.

d) Find the evolution $|\psi_i(t)\rangle$ of each element of the ensemble found in part (b). Use this result to find the density matrix $\hat{\rho}_A(t)$.

e) Find the evolution of $\hat{\rho}_A(t)$ directly from $\hat{\rho}_A(0)$ and the evolution operator. Check that your result is consistent with part (d).

f) Find the purity of the state of Alice's particle as a function of time.

g) A Stern-Gerlach measurement of $\hat{S}_y$ is performed on Alice's particle at time t. Find the probabilities of possible outcomes

- using the statistical ensemble $\{p_i, |\psi_i(t)\rangle\}$ found in part (d);
- using the density operator $\hat{\rho}_A(t)$.

Check that the results are the same.