PHYS 543 Fall 2017

Homework 3

Solutions

$\boxed{3.1}$ a) $H = -\vec{\mu}\cdot\vec{B} = -\gamma B \hat{S}_x$, where $S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\gamma = \frac{ge}{2Mc}$

Find evolution operator $e^{-\frac{i}{\hbar}Ht} = e^{i\gamma B t S_x/\hbar}$

Eigenvalues / states of S_x:

$$\hbar \to \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \qquad 0 \to \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad -\hbar \to \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

$$e^{-\frac{i}{\hbar}Ht} = e^{i\gamma B t}\frac{1}{4}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}\begin{pmatrix} 1 & \sqrt{2} & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \end{pmatrix} + e^{-i\gamma B t}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}\begin{pmatrix} 1 & -\sqrt{2} & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \cos\gamma B t + 1 & \cdot & \cdot \\ \sqrt{2}\, i \sin\gamma B t & \cdot & \cdot \\ \cos\gamma B t - 1 & \cdot & \cdot \end{pmatrix}$$ (only 1st column is of interest)

$$|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow |\psi(t)\rangle = e^{-i\frac{H}{\hbar}t}|\psi(0)\rangle = \frac{1}{2}\begin{pmatrix} \cos\gamma B t + 1 \\ \sqrt{2}\, i \sin\gamma B t \\ \cos\gamma B t - 1 \end{pmatrix}$$

b) $$\langle S_x \rangle = \frac{\hbar}{\sqrt{2}}\langle\psi(t)|\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}|\psi(t)\rangle$$

$$= \frac{\hbar}{4\sqrt{2}}\begin{pmatrix} \cos\gamma B t + 1 \\ -\sqrt{2}\, i \sin\gamma B t \\ \cos\gamma B t - 1 \end{pmatrix}^T \begin{pmatrix} \sqrt{2}\, i \sin\gamma B t \\ 2\cos\gamma B t \\ \sqrt{2}\, i \sin\gamma B t \end{pmatrix} = 0$$

$$\langle S_y \rangle = \frac{\hbar}{\sqrt{2}}\langle\psi(t)|\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}|\psi(t)\rangle$$

$$= \frac{\hbar}{4\sqrt{2}}\begin{pmatrix} \cos\gamma B t + 1 \\ -\sqrt{2}\, i \sin\gamma B t \\ \cos\gamma B t - 1 \end{pmatrix}^T \begin{pmatrix} \sqrt{2}\sin\gamma B t \\ 2i \\ -\sqrt{2}\sin\gamma B t \end{pmatrix} =$$

$$= \frac{\hbar}{4\sqrt{2}}\left(2\sqrt{2}\sin\gamma B t + 2\sqrt{2}\sin\gamma B t\right) = \hbar\sin\gamma B t$$

$$\langle S_z \rangle = \hbar\langle\psi(t)|\begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}|\psi(t)\rangle$$

$$= \frac{\hbar}{4}\begin{pmatrix} \cos\gamma B t + 1 \\ -\sqrt{2}\, i \sin\gamma B t \\ \cos\gamma B t - 1 \end{pmatrix}\begin{pmatrix} \cos\gamma B t + 1 \\ 0 \\ 1 - \cos\gamma B t \end{pmatrix}$$

$$= \frac{\hbar}{4}\left[(\cos\gamma B t + 1)^2 - (\cos\gamma B t - 1)^2\right] = \hbar\cos\gamma B t.$$

Consistent with classical precession around x axis with $\Omega_L = \gamma B$.

c) $\left| \langle m_y = 1 \mid \psi(t) \rangle \right|^2 = \left| \frac{1}{4} \begin{pmatrix} 1 \\ -\sqrt{2}i \\ -1 \end{pmatrix}^T \begin{pmatrix} \cos\gamma Bt + 1 \\ \sqrt{2}i \sin\gamma Bt \\ \cos\gamma Bt - 1 \end{pmatrix} \right|^2$

$= \left| \frac{1}{4} (2 + 2\sin\gamma Bt) \right|^2 = \frac{1}{4} (1 + \sin\gamma Bt)^2 \quad \left(= 1 \text{ for } \gamma Bt = \frac{\pi}{2},\ 0 \text{ for } \frac{3\pi}{2} \right)$

$\left| \langle m_y = 0 \mid \psi(t) \rangle \right|^2 = \left| \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ +1 \end{pmatrix} \begin{pmatrix} \cos\gamma Bt + 1 \\ \sqrt{2}i \sin\gamma Bt \\ \cos\gamma Bt - 1 \end{pmatrix} \right|^2$

$= \left| \frac{1}{2\sqrt{2}} \, 2\cos\gamma Bt \right|^2 = \frac{1}{2} \cos^2 \gamma Bt$

$\left| \langle m_y = -1 \mid \psi(t) \rangle \right|^2 = \left| \frac{1}{4} \begin{pmatrix} 1 \\ \sqrt{2}i \\ -1 \end{pmatrix} \begin{pmatrix} \cos\gamma Bt + 1 \\ \sqrt{2}i \sin\gamma Bt \\ \cos\gamma Bt - 1 \end{pmatrix} \right|^2$

$= \left| \frac{1}{4} (2 - 2\sin\gamma Bt) \right|^2 = \frac{1}{4} (1 - \sin\gamma Bt)^2 \quad \left(= 0 \text{ for } \gamma Bt = \frac{\pi}{2},\ 1 \text{ for } \frac{3\pi}{2} \right)$

$\sum pr = 1.$

$\boxed{3.2}$

a) $\frac{1}{\sqrt{2}}\begin{pmatrix}\cos\theta/2 & i\sin\theta/2\\ i\sin\theta/2 & \cos\theta/2\end{pmatrix}\begin{pmatrix}e^{-i\Delta t/2} & \\ & e^{i\Delta t/2}\end{pmatrix}\begin{pmatrix}\cos\pi/4 & i\sin\pi/4\\ i\sin\pi/4 & \cos\pi/4\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$

$=\frac{1}{\sqrt{2}}\begin{pmatrix}\cos\theta/2 & i\sin\theta/2\\ i\sin\theta/2 & \cos\theta/2\end{pmatrix}\begin{pmatrix}e^{-i\Delta t/2}\\ ie^{i\Delta t/2}\end{pmatrix}$

$=\frac{1}{\sqrt{2}}\begin{pmatrix}\cos\theta/2\, e^{-i\Delta t/2} - i\sin\theta/2\, e^{i\Delta t/2}\\ i\sin\theta/2\, e^{-i\Delta t/2} + i\cos\theta/2\, e^{i\Delta t/2}\end{pmatrix} \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}\cos\theta/2 - \sin\theta/2\, e^{i\Delta t}\\ i\sin\theta/2 + i\cos\theta/2\, e^{i\Delta t}\end{pmatrix}$

$pr_{\uparrow} = \frac{1}{2}\left[(\cos\theta/2 - \sin\theta/2\cos\Delta t)^2 + (\sin\theta/2\sin\Delta t)^2\right]$

$= \frac{1}{2}\left[\cos^2\theta/2 + \sin^2\theta/2 - 2\sin\theta/2\cos\theta/2\cos\Delta t\right] = \frac{1}{2}(1 - \sin\theta\cos\Delta t)$

$pr_{\downarrow} = \frac{1}{2}(1 + \sin\theta\cos\Delta t)$

b) $\frac{1}{\sqrt{2}}\begin{pmatrix}\cos\pi/4 & i\sin\pi/4\\ i\sin\pi/4 & \cos\pi/4\end{pmatrix}\begin{pmatrix}e^{-i\Delta t/2} & \\ & e^{i\Delta t/2}\end{pmatrix}\begin{pmatrix}\cos\theta/2 & i\sin\theta/2\\ i\sin\theta/2 & \cos\theta/2\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$

$=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}\cos\theta/2\, e^{-i\Delta t/2}\\ i\sin\theta/2\, e^{i\Delta t/2}\end{pmatrix}$

$=\frac{1}{\sqrt{2}}\begin{pmatrix}\cos\theta/2\, e^{-i\Delta t/2} - \sin\theta/2\, e^{i\Delta t/2}\\ i\cos\theta/2\, e^{i\Delta t/2} + i\sin\theta/2\, e^{i\Delta t/2}\end{pmatrix} \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}\cos\theta/2 - \sin\theta/2\, e^{i\Delta t}\\ i\cos\theta/2\, e^{i\Delta t} + i\sin\theta/2\end{pmatrix}$

$pr_{\uparrow} = \frac{1}{2}(\cos\theta/2 - \sin\theta/2\cos\Delta t)^2 + (\sin\theta/2\sin\Delta t)^2 = \frac{1}{2}(1 - \sin\theta\cos\Delta t)$

$pr_{\downarrow} = \frac{1}{2}(1 + \sin\theta\cos\Delta t)$

$\boxed{3}$ $|l=2, m=2\rangle = |m_1 = 3/2, m_2 = 1/2\rangle$ (1)

$$\Rightarrow \boxed{\langle 2,2|3/2\,;1/2\rangle = 1}$$

Apply $L_- = L_{1-} + L_{2-}$ to both sides of (1)

$$\sqrt{2(2+1)-2(2-1)}\,|2\;1\rangle = \sqrt{\tfrac32(\tfrac32+1)-\tfrac32(\tfrac32-1)}\,|\tfrac12\;\tfrac12\rangle + \sqrt{\tfrac12(\tfrac12+1)-\tfrac12(\tfrac12-1)}\,|\tfrac32\;-\tfrac12\rangle$$

$$|2\;1\rangle = \frac{\sqrt3}{2}|\tfrac12\;\tfrac12\rangle + \frac12|\tfrac32\;-\tfrac12\rangle \qquad (2)$$

$$\Rightarrow \boxed{\langle 2,1|\tfrac12\;\tfrac12\rangle = \sqrt{\tfrac34}}\;;\; \boxed{\langle 2\;1|\tfrac32,-\tfrac12\rangle = \tfrac12}$$

Apply L_- again to (2)

$$\sqrt{2(2+1)-1(1-1)}\,|2\;0\rangle = \frac{\sqrt3}{2}\sqrt{\tfrac32(\tfrac32+1)-\tfrac12(\tfrac12-1)}\,|-\tfrac12\;\tfrac12\rangle + \frac{\sqrt3}{2}\sqrt{\tfrac12(\tfrac12+1)-\tfrac12(\tfrac12-1)}\,|\tfrac12\;-\tfrac12\rangle + \frac12\sqrt{\tfrac32(\tfrac32+1)-\tfrac32(\tfrac32-1)}\,|\tfrac12\;-\tfrac12\rangle$$

$$\sqrt6\,|2\;0\rangle = \sqrt3\,|-\tfrac12\;\tfrac12\rangle + \sqrt3\,|\tfrac12\;-\tfrac12\rangle$$

$$|2\;0\rangle = \frac{1}{\sqrt2}|-\tfrac12\;\tfrac12\rangle + \frac{1}{\sqrt2}|\tfrac12\;-\tfrac12\rangle \qquad (3)$$

$$\Rightarrow \boxed{\langle 2,0|-\tfrac12\;\tfrac12\rangle = \tfrac{1}{\sqrt2}}\;;\; \boxed{\langle 2\,0|\tfrac12\;-\tfrac12\rangle = +\tfrac{1}{\sqrt2}}$$

Apply L_- again to (3)

$$\sqrt{2(2+1)-0(0-1)}\,|2\;-1\rangle = \frac{1}{\sqrt2}\sqrt{\tfrac32(\tfrac32+1)-(-\tfrac12)(-\tfrac12-1)}\,|-\tfrac32\;\tfrac12\rangle + \frac{1}{\sqrt2}\sqrt{\tfrac12(\tfrac12+1)-\tfrac12(\tfrac12-1)}\,|-\tfrac12\;-\tfrac12\rangle + \frac{1}{\sqrt2}\sqrt{\tfrac32(\tfrac32+1)-\tfrac12(\tfrac12-1)}\,|-\tfrac12\;-\tfrac12\rangle$$

$$\sqrt6\,|2\;-1\rangle = \frac{1}{\sqrt2}\sqrt3\,|-\tfrac32\;\tfrac12\rangle + \frac{1}{\sqrt2}3\,|-\tfrac12\;-\tfrac12\rangle$$

$$|2\;-1\rangle = \frac12|-\tfrac32\;\tfrac12\rangle + \frac{\sqrt3}{2}|-\tfrac12\;-\tfrac12\rangle \qquad (4)$$

$$\Rightarrow \boxed{\langle 2\;-1|-\tfrac32\;\tfrac12\rangle = \tfrac12}\;;\; \boxed{\langle 2\;-1|-\tfrac12\;-\tfrac12\rangle = \sqrt{\tfrac34}}$$

$\boxed{4}$ $|\ell=2, m=-1\rangle = \frac{1}{\sqrt{2}}|0\ \ -1\rangle + \frac{1}{\sqrt{2}}|-1\ \ 0\rangle$

$|m_x = 1\rangle = \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-1\rangle$

$\langle m_x = 1_{Alice} | \ell = 2, m = -1\rangle = \frac{1}{2}|-1\rangle_{Bob} + \frac{1}{2\sqrt{2}}|0\rangle_{Bob}$

probability of event $= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2 = \frac{3}{8}$