PHYS 543 Fall 2017

Homework 2

Solutions

$\boxed{2.1}$ a) $(L^2)_{mm'} = \hbar^2 \ell(\ell+1)\,\delta_{mm'} = \frac{15}{4}\hbar^2 \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$$(L_z)_{mm'} = m\hbar\,\delta_{mm'} = \hbar \begin{pmatrix} 3/2 & & & \\ & 1/2 & & \\ & & -1/2 & \\ & & & -3/2 \end{pmatrix}$$

$$(L_+)_{mm'} = \hbar\sqrt{\frac{15}{4} - m'(m'+1)}\;\delta_{m,m'+1} = \hbar \begin{pmatrix} 0 & \sqrt{3} & & \\ & 0 & 2 & \\ & & 0 & \sqrt{3} \\ & & & 0 \end{pmatrix}$$

$$(L_+)_{3/2,\,1/2} = \sqrt{\frac{15}{4} - \frac{3}{4}} = \sqrt{3}$$

$$(L_+)_{1/2,\,-1/2} = \sqrt{\frac{15}{4} + \frac{1}{4}} = 2$$

$$(L_+)_{-1/2,\,-3/2} = \sqrt{\frac{15}{4} - \frac{3}{4}} = \sqrt{3}$$

$$(L_-)_{mm'} = \hbar\sqrt{\frac{15}{4} - m'(m'-1)}\;\delta_{m,m'-1} = \hbar \begin{pmatrix} 0 & & & \\ \sqrt{3} & 0 & & \\ & 2 & 0 & \\ & & \sqrt{3} & 0 \end{pmatrix}$$

$$L_x = (L_+ + L_-)/2 = \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{3} & & \\ \sqrt{3} & 0 & 2 & \\ & 2 & 0 & \sqrt{3} \\ & & \sqrt{3} & 0 \end{pmatrix}$$

$$L_y = (L_+ - L_-)/2i = \frac{\hbar}{2} \begin{pmatrix} 0 & -\sqrt{3}i & & \\ \sqrt{3}i & 0 & -2i & \\ & 2i & 0 & -\sqrt{3}i \\ & & \sqrt{3}i & 0 \end{pmatrix}$$

b) $$L_x^2 = \frac{\hbar^2}{4}\begin{pmatrix} 3 & & 2\sqrt{3} & \\ & 7 & & 2\sqrt{3} \\ 2\sqrt{3} & & 7 & \\ & 2\sqrt{3} & & 3 \end{pmatrix}$$

$$L_y^2 = \frac{\hbar^2}{4}\begin{pmatrix} 3 & & -2\sqrt{3} & \\ & 7 & & -2\sqrt{3} \\ -2\sqrt{3} & & 7 & \\ & -2\sqrt{3} & & 3 \end{pmatrix}$$

$$L_z^2 = \frac{\hbar^2}{4}\begin{pmatrix} 9 & & & \\ & 1 & & \\ & & 1 & \\ & & & 9 \end{pmatrix}$$

$$L_x^2 + L_y^2 + L_z^2 = \frac{\hbar^2}{4}\begin{pmatrix} 15 & & & \\ & 15 & & \\ & & 15 & \\ & & & 15 \end{pmatrix} \quad \checkmark$$

c) $$[L_x\ L_y] = \frac{\hbar^2}{4}\begin{pmatrix} 3i & & -2i\sqrt{3} & \\ & i & & -2i\sqrt{3} \\ 2i\sqrt{3} & & -i & \\ & 2i\sqrt{3} & & -3i \end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix} -3i & & -2i\sqrt{3} & \\ & -i & & -2i\sqrt{3} \\ 2i\sqrt{3} & & +i & \\ & 2i\sqrt{3} & & +3i \end{pmatrix} =$$

$$= \frac{i\hbar^2}{2}\begin{pmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & -3 \end{pmatrix} = i\hbar L_z \quad \checkmark$$

$$[L_x\ L_z] = \frac{\hbar^2}{4}\begin{pmatrix} 0 & \sqrt{3} & & \\ 3\sqrt{3} & 0 & -2 & \\ & 2 & 0 & 3\sqrt{3} \\ & & -\sqrt{3} & \end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix} 0 & 3\sqrt{3} & & \\ \sqrt{3} & 0 & 2 & \\ & -2 & 0 & -\sqrt{3} \\ & & -3\sqrt{3} & 0 \end{pmatrix}$$

$$= \hbar^2\begin{pmatrix} 0 & -\sqrt{3} & & \\ \sqrt{3} & 0 & -2 & \\ & 2 & 0 & -\sqrt{3} \\ & & +\sqrt{3} & 0 \end{pmatrix} = -i\hbar L_y \quad \checkmark$$

$$[L_y\ L_z] = \frac{\hbar^2}{4}\begin{pmatrix} 0 & \sqrt{3}i & & \\ 3\sqrt{3}i & 0 & +2i & \\ & 2i & 0 & 3\sqrt{3}i \\ & & -\sqrt{3}i & 0 \end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix} 0 & -3\sqrt{3}i & & \\ \sqrt{3}i & 0 & -2i & \\ & -2i & 0 & \sqrt{3}i \\ & & -3\sqrt{3}i & \end{pmatrix}$$

$$= \hbar^2\begin{pmatrix} 0 & \sqrt{3}i & & \\ \sqrt{3}i & 0 & 2i & \\ & 2i & 0 & \sqrt{3}i \\ & & \sqrt{3}i & 0 \end{pmatrix} = i\hbar L_x$$

d) Eigenstates of L_x:

$$|+\tfrac{3}{2}\rangle = \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\ \sqrt{3}\\ \sqrt{3}\\ 1\end{pmatrix};\quad |+\tfrac{1}{2}\rangle = \frac{1}{2\sqrt{2}}\begin{pmatrix}-\sqrt{3}\\ -1\\ 1\\ \sqrt{3}\end{pmatrix};\quad |-\tfrac{1}{2}\rangle = \frac{1}{2\sqrt{2}}\begin{pmatrix}\sqrt{3}\\ -1\\ -1\\ \sqrt{3}\end{pmatrix};\quad |-\tfrac{3}{2}\rangle = \frac{1}{2\sqrt{2}}\begin{pmatrix}-1\\ \sqrt{3}\\ -\sqrt{3}\\ 1\end{pmatrix}$$

e)
$$L_{x+} = L_y + iL_z = \frac{\hbar}{2}\begin{pmatrix}3i & -\sqrt{3}i & 0 & 0\\ \sqrt{3}i & i & -2i & 0\\ 0 & 2i & -i & -\sqrt{3}i\\ 0 & 0 & \sqrt{3}i & -3i\end{pmatrix}$$

$$L_{x-} = L_{x+}^{\dagger} = \frac{\hbar}{2}\begin{pmatrix}-3i & -\sqrt{3}i & 0 & 0\\ \sqrt{3}i & -i & -2i & 0\\ 0 & 2i & i & -\sqrt{3}i\\ 0 & 0 & \sqrt{3}i & 3i\end{pmatrix}$$

f)
$$L_{x+}|-\tfrac{3}{2}\rangle = \frac{\hbar}{2}\begin{pmatrix}-6i\\ +2\sqrt{3}i\\ 2\sqrt{3}i\\ -6i\end{pmatrix} = \underbrace{\sqrt{\frac{3}{2}\frac{5}{2} - \left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)}}_{\sqrt{3}}\hbar\begin{pmatrix}\sqrt{3}\\ -1\\ -1\\ \sqrt{3}\end{pmatrix}(-i)$$

$$L_{x+}|-\tfrac{1}{2}\rangle = \frac{\hbar}{2}\begin{pmatrix}4\sqrt{3}i\\ 4i\\ -4i\\ -4\sqrt{3}i\end{pmatrix} = \underbrace{\sqrt{\frac{3}{2}\frac{5}{2} - \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)}}_{2}\hbar\begin{pmatrix}-\sqrt{3}\\ -1\\ 1\\ \sqrt{3}\end{pmatrix}(-i)$$

$$L_{x+}|\tfrac{1}{2}\rangle = \frac{\hbar}{2}\begin{pmatrix}2\sqrt{3}i\\ 6i\\ -6i\\ -2\sqrt{3}i\end{pmatrix} = \underbrace{\sqrt{\frac{3}{2}\frac{5}{2} - \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}}_{\sqrt{3}}\begin{pmatrix}1\\ \sqrt{3}\\ \sqrt{3}\\ 1\end{pmatrix}(i)$$

$$L_{x+}|\tfrac{3}{2}\rangle = \frac{\hbar}{2}\begin{pmatrix}0\\ 0\\ 0\\ 0\end{pmatrix}$$

$\boxed{2.2}$ Work in $|m_1, m_2\rangle$ basis. State in question: $|\psi\rangle$

$$L_z|\psi\rangle = (L_{1z} + L_{2z})|m_1 = \ell_1, m_2 = \ell_2\rangle = \hbar(m_1 + m_2)|\psi\rangle$$

$$L^2|\psi\rangle = (L_1^2 + L_2^2 + 2L_1 L_2)|\psi\rangle$$

$$= (L_1^2 + L_2^2 + 2L_{1z}L_{2z} + \underbrace{L_{1+}L_{2-} + L_{1-}L_{2+}}_{\text{yield 0 acting on } |\psi\rangle})|\psi\rangle$$

because $L_{1+}|\ell_1\,\ell_1\rangle = L_{2+}|\ell_2\,\ell_2\rangle = 0$

$$= [\hbar^2 \ell_1(\ell_1+1) + \hbar^2 \ell_2(\ell_2+1) + 2\hbar^2 \ell_1 \ell_2]|\psi\rangle$$

$$= \hbar^2(\ell_1^2 + \ell_2^2 + \ell_1 + \ell_2 + 2\ell_1\ell_2)|\psi\rangle$$

$$= \hbar^2(\ell_1 + \ell_2)(\ell_1 + \ell_2 + 1)|\psi\rangle$$

$\boxed{2.3}$ To change from "old" to "new" reference frame, we need to (1) rotate around the z axis by φ and (2) rotate around the new y axis by $-\Theta$. The coordinates are then related according to

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & \\ \sin\varphi & \cos\varphi & \\ & & 1 \end{pmatrix} \begin{pmatrix} \cos\Theta & & \sin\Theta \\ & 1 & \\ -\sin\Theta & & \cos\Theta \end{pmatrix} \begin{pmatrix} L_x' \\ L_y' \\ L_z' \end{pmatrix}$$

or

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} \cos\Theta\cos\varphi & -\sin\varphi & \sin\Theta\cos\varphi \\ \cos\Theta\sin\varphi & \cos\varphi & \sin\Theta\sin\varphi \\ -\sin\Theta & 0 & \cos\Theta \end{pmatrix} \begin{pmatrix} L_x' \\ L_y' \\ L_z' \end{pmatrix}$$

Hence $\langle L_x \rangle = \cos\Theta \sin\varphi \langle L_x' \rangle - \sin\varphi \langle L_y' \rangle + \sin\Theta\cos\varphi \langle L_z' \rangle$

$$= 0 + 0 + \sin\Theta\cos\varphi\,(\hbar m)$$

because $\left.\begin{array}{l} \langle L_x' \rangle = \langle L_y' \rangle = 0 \\ \langle L_z' \rangle = m\hbar \end{array}\right\}$ in state $|l\, m_{\Theta\varphi}\rangle$

Similarly,

$$\langle L_y \rangle = \sin\Theta \sin\varphi\,(\hbar m)$$

$$\langle L_z \rangle = \cos\Theta\,(\hbar m)$$

$\boxed{25}$ $\langle n\ell m| \begin{pmatrix} x \\ y \\ z \end{pmatrix} |n'\ell' m'\rangle$

$$= \int_0^\infty \int_0^\pi \int_0^{2\pi} r^2 \sin\Theta\, R^*_{n\ell}(r) R_{n'\ell'}(r)\, Y_\ell^m(\Theta\varphi)^+ Y_{\ell'}^{m'}(\Theta\varphi) \begin{pmatrix} r\sin\Theta\cos\varphi \\ r\sin\Theta\sin\varphi \\ r\cos\Theta \end{pmatrix} d\varphi d\Theta$$

$$= \underbrace{\left(\int_0^\infty r^3 R^*_{n\ell}(r) R_{n'\ell'}(r)\, dr\right)}_{\equiv I_R} \underbrace{\left(\int_0^\pi\int_0^{2\pi} \sin\Theta\, Y_\ell^m(\Theta\varphi)\, Y_{\ell'}^{m'}(\Theta\varphi) \begin{pmatrix} \sin\Theta\cos\varphi \\ \sin\Theta\sin\varphi \\ \cos\Theta \end{pmatrix} d\varphi d\Theta\right)}_{\equiv I_Y \begin{pmatrix} x \\ y \\ z \end{pmatrix}}$$

a) For $\ell = m = \ell' = m' = 0$, $Y_\ell^m = Y_{\ell'}^{m'} = \sqrt{\frac{1}{4\pi}} \Rightarrow I_y = 0$

b)
$$Y_{00} = \sqrt{\frac{1}{4\pi}} \quad ; \quad Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\Theta$$

do not depend on $\varphi \Rightarrow I_Y(x) = I_Y(y) = 0$

$$I_Y(z) = \sqrt{3}\,\frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \sin\Theta\cos^2\Theta\, d\varphi\, d\Theta$$

$$= \frac{\sqrt{3}}{2} \int_0^\pi \sin\Theta \cos^2\Theta\, d\Theta$$

$\boxed{\begin{aligned} x &= \cos\Theta \\ dx &= -\sin\Theta\, d\Theta \end{aligned}}$

$$= -\frac{\sqrt{3}}{2} \int_1^{-1} x^2\, dx = +\frac{\sqrt{3}}{2}\,\frac{2}{3} = \frac{1}{\sqrt{3}}$$

$$I_R = \int_0^\infty r^3 R_{10}(r)\, R_{21}(r)\, dr$$

$$= 2\,\frac{1}{\sqrt{24}}\, a^{-4} \int r^4 e^{-r/2a} e^{-r/a}\, dr = \frac{1}{\sqrt{6}}\, a^{-4} \int r^4 e^{-3r/2a} dr$$

$$= \frac{1}{\sqrt{6}}\,\frac{256}{81}\, a$$

$$\langle 1\,0\,0 | z | 2\,1\,0\rangle = I_Y(z)\, I_R = \frac{1}{\sqrt{2}}\,\frac{256}{243}\, a$$

c) $Y_{00} = \sqrt{\frac{1}{4\pi}}$; $Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi}$

$$I_Y(x) = -\sqrt{\frac{3}{2}} \frac{1}{4\pi} \int_0^{\pi}\int_0^{2\pi} \sin^3\theta \cos\varphi e^{i\varphi} \, d\varphi \, d\theta$$

$$= -\sqrt{\frac{3}{2}} \frac{1}{4\pi} \int_0^{\pi} \sin^3\theta \, d\theta \int_0^{2\pi} \cos^2\varphi \, d\varphi$$

$$= -\sqrt{\frac{3}{2}} \frac{1}{4\pi} \frac{4}{3} \pi = -\sqrt{\frac{1}{6}}$$

$$I_Y(y) = -\sqrt{\frac{3}{2}} \frac{1}{4\pi} \int_0^{\pi}\int_0^{2\pi} \sin^3\theta \sin\varphi e^{i\varphi} \, d\varphi \, d\theta$$

$$= -i\sqrt{\frac{3}{2}} \frac{1}{4\pi} \int_0^{\pi} \sin^3\theta \, d\theta \int_0^{2\pi} \sin^2\varphi \, d\varphi = -i\sqrt{\frac{1}{6}}$$

$I_R = \frac{1}{\sqrt{6}} \frac{256}{81} a$ (same as in (b))

$$\Rightarrow \langle 1\,0\,0 \left| \begin{matrix} x \\ y \\ z \end{matrix} \right| 2\,1\,1 \rangle = \begin{pmatrix} -128/243 \\ -128i/243 \\ 0 \end{pmatrix}$$

2.4 Follow the logic of Ex. 4.38 and 4.43.

Because $\ell+1 \le j \le n$, only A_n and A_{n-1} survive.

They are related by (use $\varkappa = \frac{1}{na}$)

$$\left(2\varkappa(n-1) - \frac{2}{a}\right)A_{n-1} + \left((n-2)(n-1) - (n-1)n\right)A_n = 0$$

$$\frac{2}{a}\left(-\frac{1}{n}\right)A_{n-1} - 2(n-1)A_n = 0$$

$$A_n = -\frac{1}{an(n-1)}A_{n-1}$$

Hence $R(r) = N\left(1 - \frac{1}{n(n-1)}\frac{r}{a}\right)\left(\frac{r}{a}\right)^{n-2} e^{-r/na}$

Find normalization factor.

$$\int_0^\infty R^2(r)\, r^2 dr = 1$$

$$N^2 a^2 \int_0^\infty \left[\left(\frac{r}{a}\right)^{2n-2} - \frac{2}{n(n-1)}\left(\frac{r}{a}\right)^{2n-1} + \frac{1}{n^2(n-1)^2}\left(\frac{r}{a}\right)^{2n}\right] e^{-2r/na} dr = 1$$

Change integration variable

$$x = \frac{2r}{na}. \quad \text{Then} \quad \frac{r}{a} = \frac{n}{2}x \quad ; \quad dr = \frac{na}{2}dx$$

$$N^2 a^3 \left(\frac{na}{2}\right)\left(\frac{n}{2}\right)^{2n-2} \int_0^\infty \left[x^{2n-2} - \frac{1}{(n-1)}x^{2n-1} + \frac{1}{4(n-1)^2}x^{2n}\right] e^{-x} dx = 1$$

$$N^2 a^3 \left(\frac{n}{2}\right)^{2n-1}\left[(2n-2)! - \frac{(2n-1)!}{n-1} + \frac{(2n)!}{4(n-1)^2}\right] = 1$$

$$N^2 a^3 \left(\frac{n}{2}\right)^{2n-1}(2n-2)!\left[1 - \frac{2n-1}{n-1} + \frac{2n(2n-1)}{4(n-1)^2}\right] = 1$$

$$N^2 a^3 \left(\frac{n}{2}\right)^{2n-1}(2n-2)!\left[\frac{4(n-1)^2 - 4(2n-1)(n-1) + 2n(2n-1)}{4(n-1)^2}\right] = 1$$

$$N^2 a^3 \left(\frac{n}{2}\right)^{2n-1}(2n-2)!\,\frac{2n}{4(n-1)^2} = 1$$

$$N^2 a^3 \left(\frac{n}{2}\right)^{2n-1}\frac{(2n-3)!\,n}{n-1} = 1$$

$$N = a^{-3/2}\left(\frac{2}{n}\right)^n \sqrt{\frac{n-1}{2(2n-3)!}}$$

2.5 $\langle n\, l\, m | \begin{pmatrix} x \\ y \\ z \end{pmatrix} | n'\, l'\, m' \rangle$

$$= \int_0^\infty \int_0^\pi \int_0^{2\pi} r^2 \sin\Theta\, R^*_{nl}(r) R_{n'l'}(r)\, Y_l^m(\Theta\,\varphi)^+\, Y_{l'}^{m'}(\Theta\,\varphi) \begin{pmatrix} r\sin\Theta\cos\varphi \\ r\sin\Theta\sin\varphi \\ r\cos\Theta \end{pmatrix} d\varphi\, d\Theta$$

$$= \underbrace{\left(\int_0^\infty r^3 R^*_{nl}(r) R_{n'l'}(r)\, dr\right)}_{\equiv I_R} \underbrace{\left(\int_0^\pi\int_0^{2\pi} \sin\Theta\, Y_l^m(\Theta\,\varphi)\, Y_{l'}^{m'}(\Theta\,\varphi) \begin{pmatrix} \sin\Theta\cos\varphi \\ \sin\Theta\sin\varphi \\ \cos\Theta \end{pmatrix} d\varphi\, d\Theta\right)}_{\equiv I_Y \begin{pmatrix} x \\ y \\ z \end{pmatrix}}$$

a) For $l = m = l' = m' = 0$, $Y_l^m = Y_{l'}^{m'} = \sqrt{\frac{1}{4\pi}} \Rightarrow I_y = 0$

b) $Y_{00} = \sqrt{\frac{1}{4\pi}}$; $Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\Theta$

do not depend on $\varphi \Rightarrow I_Y(x) = I_Y(y) = 0$

$$I_Y(z) = \sqrt{3}\, \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \sin\Theta \cos^2\Theta\, d\varphi\, d\Theta$$

$$= \frac{\sqrt{3}}{2} \int_0^\pi \sin\Theta \cos^2\Theta\, d\Theta \qquad \boxed{\begin{array}{l} x = \cos\Theta \\ dx = -\sin\Theta\, d\Theta \end{array}}$$

$$= -\frac{\sqrt{3}}{2} \int_1^{-1} x^2\, dx = +\frac{\sqrt{3}}{2}\, \frac{2}{3} = \frac{1}{\sqrt{3}}$$

$$I_R = \int_0^\infty r^3 R_{10}(r)\, R_{21}(r)\, dr$$

$$= 2 \frac{1}{\sqrt{24}} a^{-4} \int r^4 e^{-r/2a} e^{-r/a} dr = \frac{1}{\sqrt{6}} a^{-4} \int r^4 e^{-3r/2a} dr$$

$$= \frac{1}{\sqrt{6}}\, \frac{256}{81}\, a$$

$$\langle 1\,0\,0 | z | 2\,1\,0 \rangle = I_Y(z)\, I_R = \frac{1}{\sqrt{2}}\, \frac{256}{243}\, a$$

c) $Y_{00} = \sqrt{\frac{1}{4\pi}}$; $Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\Theta e^{i\varphi}$

$$I_{\varphi}(x) = -\sqrt{\frac{3}{2}} \frac{1}{4\pi} \int_0^{\pi}\int_0^{2\pi} \sin^3\Theta \cos\varphi e^{i\varphi} d\varphi d\Theta$$

$$= -\sqrt{\frac{3}{2}} \frac{1}{4\pi} \int_0^{\pi} \sin^3\Theta \, d\Theta \int_0^{2\pi} \cos^2\varphi \, d\varphi$$

$$= -\sqrt{\frac{3}{2}} \frac{1}{4\pi} \frac{4}{3} \pi = -\sqrt{\frac{1}{6}}$$

$$I_{\varphi}(y) = -\sqrt{\frac{3}{2}} \frac{1}{4\pi} \int_0^{\pi}\int_0^{2\pi} \sin^3\Theta \sin\varphi e^{i\varphi} d\varphi d\Theta$$

$$= -i\sqrt{\frac{3}{2}} \frac{1}{4\pi} \int_0^{\pi} \sin^3\Theta \, d\Theta \int_0^{2\pi} \sin^2\varphi \, d\varphi = -i\sqrt{\frac{1}{6}}$$

$I_R = \frac{1}{\sqrt{6}} \frac{256}{81} a$ (same as in (8))

$$\Rightarrow \langle 1\,0\,0 \left| \begin{matrix} x \\ y \\ z \end{matrix} \right| 2\,1\,1 \rangle = \begin{pmatrix} -128/243 \\ -128i/243 \\ 0 \end{pmatrix}$$

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a) A QWP @ 0° has operator $i|H\rangle\langle H| + |V\rangle\langle V| = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$

Input states: $\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$

Output states. $\begin{pmatrix} \cos\alpha \\ i\sin\alpha \end{pmatrix}$

Bloch. $\theta = 2\alpha$, $\varphi = \pi/2$ Meridian through x axis

b) QWP @ 45° has operator $i|+\rangle\langle+| + |-\rangle\langle-|$

$$= \frac{i}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ -1+i & 1+i \end{pmatrix}$$

Output states. $\begin{pmatrix} (\cos\alpha - \sin\alpha) + i(\cos\alpha + \sin\alpha) \\ (-\cos\alpha + \sin\alpha) + i(\cos\alpha + \sin\alpha) \end{pmatrix} = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}$

Notice that.

- $|\psi_\uparrow| = |\psi_\downarrow| = 1/2$
- $\arg \psi_\uparrow = \frac{\pi}{2} - \arg \psi_\downarrow$
- $\arg \psi_\uparrow$ takes all possible values from 0 to 2π

$\Rightarrow$ trajectory is the equator.