PHYS 543 Fall 2017

Homework 1

Solutions

$\boxed{1.1}$ a) $\mathcal{N}^2 \left(\langle\alpha| a\right)\left(a^+|\alpha\rangle\right) = 1$

$\mathcal{N}^2 \langle\alpha|(a^+a+1)|\alpha\rangle = 1$

$\mathcal{N}^2 (|\alpha|^2+1) = 1$

$\mathcal{N} = \frac{1}{\sqrt{|\alpha|^2+1}}$

b) $\alpha = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$

$\mathcal{N} a^+ |\alpha\rangle = \mathcal{N} e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n \sqrt{n+1}}{\sqrt{n!}} |n+1\rangle$

c) $\langle x\rangle = \mathcal{N}^2 \left(\langle\alpha|\hat{a}\,\hat{x}\,\hat{a}^+|\alpha\rangle\right) = \frac{1}{\sqrt{2}} \mathcal{N}^2 \langle\alpha| a(a+a^+)a^+ |\alpha\rangle$

$$a(a+a^+)a^+ = aaa^+ + aa^+a^+$$
$$= a(a^+a+1) + (a^+a+1)a^+$$
$$= aa^+a + a + a^+aa^+ + a^+$$
$$= a^+aa + 2a + a^+a^+a + 2a^+$$

$\Rightarrow \langle x\rangle = \frac{1}{\sqrt{2}} \mathcal{N}^2 \left(|\alpha|^2\alpha + 2\alpha + |\alpha|\alpha^* + 2\alpha^*\right)$

$= \frac{1}{\sqrt{2}} \frac{(|\alpha|^2+2)(\alpha+\alpha^*)}{|\alpha|^2+1} = \frac{|\alpha|^2+2}{|\alpha|^2+1} \operatorname{Re}\alpha \sqrt{2}$

d) $\langle x|\alpha\rangle = \Psi_0(x-\alpha\sqrt{2}) = \frac{1}{\pi^{1/4}} e^{-(x-\alpha\sqrt{2})^2/2}$

$\mathcal{N}\langle x|a^+|\alpha\rangle = \mathcal{N}\langle x| \frac{x-ip}{\sqrt{2}} |\alpha\rangle$

$= \frac{\mathcal{N}}{\pi^{1/4}\sqrt{2}} \left(x - \frac{d}{dx}\right) e^{-(x-\alpha\sqrt{2})^2/2}$

$= \frac{1}{\pi^{1/4}\sqrt{2(|\alpha|^2+1)}} (2x - \alpha\sqrt{2}) e^{-(x-\alpha\sqrt{2})^2/2}$

e) $\alpha \to 0 \qquad a^+|\alpha\rangle \to |1\rangle$

$\alpha \to \infty \qquad a^+|\alpha\rangle \to |\alpha\rangle$

f) $|\psi\rangle = \hat{D}\,(c|0\rangle + d|1\rangle)$ (denote $\hat{D}_x(\alpha\sqrt{2}) \equiv \hat{D}$)

$= c\hat{D}|0\rangle + d\hat{D}\hat{a}^+|0\rangle$ (use $\hat{a}^+\hat{D} = \hat{D}\hat{a}^+ + \alpha\hat{D}$)

$= c\hat{D}|0\rangle + d\hat{a}^+\hat{D}|0\rangle - \alpha d\hat{D}|0\rangle$

$= (c-\alpha d)|\alpha\rangle + d\,a^+|\alpha\rangle$

$= d\,a^+|\alpha\rangle$ if $c - \alpha d = 0$

$\Downarrow$

$c = \frac{-\alpha}{\sqrt{1+\alpha^2}}$, $d = \frac{1}{\sqrt{1+\alpha^2}}$ (normalized)

Then $|\psi\rangle = \frac{a^+|\alpha\rangle}{\sqrt{1+\alpha^2}}$ consistent with (a)

$$\langle x|\psi\rangle = c\,\psi_0(x-\alpha\sqrt{2}) + d\,\psi_1(x-\alpha\sqrt{2})$$

$$= \frac{1}{\pi^{1/4}\sqrt{1+\alpha^2}}\, e^{-(x-\alpha\sqrt{2})^2}\left(+\alpha + \sqrt{2}\,(x-\alpha\sqrt{2})\right)$$

$$= \frac{1}{\pi^{1/4}\sqrt{1+\alpha^2}}\, e^{-(x-\alpha\sqrt{2})^2}\left(x\sqrt{2} - \alpha\right)$$

consistent with (d).

$\boxed{1.2}$ a) $[X_\theta, X_{\frac{\pi}{2}+\theta}] = [Xc + Ps, -Xs + Pc]$

$$= c^2[X, P] - s^2[P, X] = i$$

$$\langle \Delta X_\theta^2 \rangle \langle \Delta X^2_{\frac{\pi}{2}+\theta} \rangle \geq \frac{\hbar^2}{4}$$

b) For $\theta = 0$: $H = -\frac{\hbar\gamma}{2}[XP + PX]$

$\Rightarrow$ for arbitrary θ:

$$H = -\frac{\hbar\gamma}{2}[X_\theta X_{\frac{\pi}{2}+\theta} + X_{\frac{\pi}{2}+\theta} X_\theta] \qquad (1)$$

Then $\dot{X}_\theta = \frac{i}{\hbar}[H, X_\theta] =$

$$= \frac{-i\gamma}{2}\left(X_\theta[X_{\frac{\pi}{2}+\theta}, X_\theta] + [X_{\frac{\pi}{2}+\theta}, X_\theta] X_\theta\right)$$

$$= -\gamma X_\theta$$

$$\dot{X}_{\frac{\pi}{2}+\theta} = \frac{i}{\hbar}[H, X_{\frac{\pi}{2}+\theta}] = \gamma X_{\frac{\pi}{2}+\theta}.$$

c) $X_\theta = \frac{a + a^+}{\sqrt{2}} \frac{e^{i\theta} + e^{-i\theta}}{2} + \frac{a - a^+}{\sqrt{2}i} \frac{e^{i\theta} - e^{-i\theta}}{2i}$

$$= \frac{1}{\sqrt{2}}(a e^{-i\theta} + a^+ e^{+i\theta})$$

From (1): $\hat{H} = -\hbar\gamma(X_\theta X_{\frac{\pi}{2}+\theta} - i)$

$$= \frac{-\hbar\gamma}{2}\left[(ae^{-i\theta} + a^+ e^{i\theta})(-iae^{-i\theta} + ia^+ e^{i\theta}) - i\right]$$

$$= -\frac{\hbar\gamma}{2}\left(-ia^2 e^{-2i\theta} + ia^{+2} e^{2i\theta} \cancel{- ia^+ a + iaa^+ - i}\right)$$

$$e^{-\frac{i}{\hbar}\hat{H}t}|0\rangle \overset{\gamma t \ll 1}{=} \left(\hat{\mathbb{1}} - \frac{i}{\hbar}\hat{H}t\right)|0\rangle$$

$$= |0\rangle - \frac{\gamma t}{2} a^{+2} e^{2i\theta} |0\rangle$$

$$= |0\rangle - \frac{\gamma t}{\sqrt{2}} e^{2i\theta} |2\rangle$$

d) $\langle\Psi| X_\theta |\Psi\rangle = \langle\Psi| X_{\frac{\pi}{2}+\theta} |\Psi\rangle = 0$

because $X_\theta|\Psi\rangle$ and $X_{\frac{\pi}{2}+\theta}|\Psi\rangle$ only contain odd photon-number terms.

$$\langle\Delta X_\theta^2\rangle = \langle\Psi| X_\theta^2 |\Psi\rangle$$

$$= \left(\langle 0| - \frac{\gamma t}{\sqrt{2}} e^{-2i\theta}\langle 2|\right)\left(\frac{a^2 e^{-2i\theta} + aa^+ + a^+a + a^{+2} e^{2i\theta}}{2}\right)\left(|0\rangle - \frac{\gamma t}{\sqrt{2}} e^{2i\theta}|2\rangle\right)$$

$$\overset{\gamma t \ll 1}{=} \langle 0| \frac{aa^+}{2} |0\rangle + \frac{1}{2}\langle 0| a^2 e^{-2i\theta}\left(\frac{\gamma t}{\sqrt{2}} e^{2i\theta}\right)|2\rangle$$

$$+ \frac{1}{2}\left(-\frac{\gamma t}{\sqrt{2}} e^{-2i\theta}\right)\langle 2| a^{+2} e^{2i\theta} |0\rangle$$

$$= \frac{1}{2}(1 - 2\gamma t) \approx \frac{1}{2} e^{-2\gamma t}$$

Similar calculation for

$$\langle\Delta X_{\frac{\pi}{2}+\theta}^2\rangle = \frac{1}{2}(1 + 2\gamma t) \approx \frac{1}{2} e^{2\gamma t}$$

e) $H = \hbar\gamma\left(X_{\theta,A} X_{\frac{\pi}{2}+\theta,B} + X_{\frac{\pi}{2}+\theta,A} X_{\theta,B}\right)$

$$\dot{X}_{\theta,\pm} = \frac{i}{\hbar}\left[H, \frac{1}{\sqrt{2}}(X_{\theta,A} \pm X_{\theta,B})\right]$$

$$= \frac{i}{\hbar}\frac{\hbar\gamma}{\sqrt{2}}\left(X_{\theta,A}\left[X_{\frac{\pi}{2}+\theta,B}, \pm X_{\theta,B}\right] + \left[X_{\frac{\pi}{2}+\theta,A}, X_{\theta,A}\right] X_{\theta,B}\right)$$

$$= \frac{i\gamma}{\sqrt{2}}\left(\mp i X_{\theta,A} - i X_{\theta,B}\right) = \pm\gamma X_{\theta,\pm}$$

$$\dot{X}_{\frac{\pi}{2}+\theta,\pm} = \frac{i}{\hbar}\left[H, \frac{1}{\sqrt{2}}(X_{\frac{\pi}{2}+\theta,A} \pm X_{\frac{\pi}{2}+\theta,B})\right]$$

$$= \frac{i\gamma}{\sqrt{2}}\left(i X_{\frac{\pi}{2}+\theta,B} \pm i X_{\frac{\pi}{2}+\theta,A}\right) = \mp\gamma X_{\frac{\pi}{2}+\theta,\pm}$$

f) $H = \hbar\gamma\left(\frac{a_A e^{-i\theta} + a_A^+ e^{i\theta}}{\sqrt{2}} \cdot \frac{-ia_B e^{-i\theta} + ia_B^+ e^{i\theta}}{\sqrt{2}} + \frac{-ia_A e^{-i\theta} + ia_A^+ e^{i\theta}}{\sqrt{2}} \frac{a_B e^{-i\theta} + a_B^+ e^{i\theta}}{\sqrt{2}}\right)$

$$= i\hbar\gamma\left(-a_A a_B e^{-2i\theta} + a_A^+ a_B^+ e^{2i\theta}\right)$$

$$e^{-\frac{i}{\hbar}Ht}|00\rangle = |00\rangle + \gamma t\, a_A^+ a_B^+ e^{2i\theta}|00\rangle$$

$$= |00\rangle + \gamma t\, e^{2i\theta}|11\rangle = |\Psi\rangle$$

g) Following Ex. 3.113:

$$\langle\Psi|X_{\theta,\pm}|\Psi\rangle = 0$$

$$\langle\Delta X_{\theta\pm}^2\rangle = \langle\Psi|X_{\theta,\pm}^2|\Psi\rangle$$

$$= \left(\langle 00| + \gamma t\, e^{-2i\theta}\langle 11|\right)\frac{a_A a_A^+ + a_A^+ a_A + a_B a_B^+ + a_B^+ a_B \pm 2a_A a_B e^{-2i\theta} \pm 2a_A^+ a_B^+ e^{2i\theta}}{4}$$

$$\times\left(|00\rangle + \gamma t\, e^{2i\theta}|11\rangle\right)$$

$$= \langle 00|\frac{a_A a_A^+ + a_B a_B^+}{4}|00\rangle \pm \gamma t\, e^{-2i\theta}\langle 11|\frac{a_A^+ a_B^+ e^{+2i\theta}}{2}|00\rangle$$

$$\pm \gamma t\, e^{2i\theta}\langle 00|\frac{a_A a_B e^{-2i\theta}}{2}|11\rangle$$

$$= \frac{1}{2} \pm \gamma t \approx \frac{1}{2}e^{\pm 2\gamma t}$$

Similarly,

$$\langle\Delta X^2_{\frac{\pi}{2}+\theta,\pm}\rangle = \frac{1}{2} \pm \gamma t \approx \frac{1}{2}e^{\mp 2\gamma t}$$

$\boxed{1.3}$

$$[L_j,[L_k,L_\ell]] = [L_j, i\hbar\, \varepsilon_{k\ell m} L_m]$$
$$= (i\hbar)^2 \varepsilon_{k\ell m}\, \varepsilon_{jmn} L_n$$
$$= -\hbar^2 \varepsilon_{mk\ell}\, \varepsilon_{mnj} L_n$$
$$= \hbar^2 L_n \left(-\delta_{kn}\delta_{\ell j} + \delta_{kj}\delta_{\ell n}\right) \tag{1}$$

$[L_x,[L_y L_z]] = [L_x, i\hbar L_x] = 0$

$j=1,\ k=2,\ \ell=3 \Rightarrow$ all δ's vanish in (1)

$[L_x,[L_x,L_z]] = [L_x, -i\hbar L_y] = \hbar^2 L_z$

$j=k=1,\ \ell=3 \Rightarrow$ second term in (1) nonzero for $n=3$

$[L_x,[L_z,L_x]] = [L_x, i\hbar L_y] = -\hbar^2 L_z$

$j=\ell=1,\ k=3 \Rightarrow$ first term in (1) nonzero for $n=3$.

$\boxed{1.4}$

$$x = r\sin\Theta\cos\varphi$$
$$y = r\sin\Theta\sin\varphi$$
$$z = r\cos\Theta$$

$$\frac{\partial}{\partial\Theta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial\Theta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial\Theta} + \frac{\partial}{\partial z}\frac{\partial z}{\partial\Theta}$$

$$= r\cos\Theta\cos\varphi\frac{\partial}{\partial x} + r\cos\Theta\sin\varphi\frac{\partial}{\partial y} - r\sin\Theta\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial\varphi} = \frac{\partial}{\partial x}\frac{\partial x}{\partial\varphi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial\varphi} + \frac{\partial}{\partial z}\frac{\partial z}{\partial\varphi}$$

$$= -r\sin\Theta\sin\varphi\frac{\partial}{\partial x} + r\sin\Theta\cos\varphi\frac{\partial}{\partial y}$$

$$L_x = i\hbar\left(\sin\varphi\frac{\partial}{\partial\Theta} + \cot\Theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$$

$$= i\hbar\left(\cancel{r\cos\Theta\sin\varphi\cos\varphi\frac{\partial}{\partial x}} + r\cos\Theta\sin^2\varphi\frac{\partial}{\partial y} - r\sin\Theta\sin\varphi\frac{\partial}{\partial z}\right.$$
$$\left.\cancel{-r\frac{\cos\Theta}{\sin\Theta}\cos\varphi\sin\Theta\sin\varphi\frac{\partial}{\partial x}} + r\frac{\cos\Theta}{\cancel{\sin\Theta}}\cos^2\varphi\cancel{\sin\Theta}\frac{\partial}{\partial y}\right)$$

$$= i\hbar\left(r\cos\Theta\frac{\partial}{\partial y} - r\sin\Theta\sin\varphi\frac{\partial}{\partial z}\right)$$

$$= i\hbar\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) = y\hat{p}_z - z\hat{p}_y$$

$$L_y = i\hbar\left(-\cos\varphi\frac{\partial}{\partial\Theta} + \cot\Theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$$

$$= i\hbar\left(-r\cos\Theta\cos^2\varphi\frac{\partial}{\partial x}\cancel{-r\cos\Theta\sin\varphi\cos\varphi\frac{\partial}{\partial y}} + r\sin\Theta\cos\varphi\frac{\partial}{\partial z}\right.$$
$$\left.-r\frac{\cos\Theta}{\cancel{\sin\Theta}}\sin^2\varphi\cancel{\sin\Theta}\frac{\partial}{\partial x} + \cancel{r\frac{\cos\Theta}{\sin\Theta}\sin\Theta\sin\varphi\cos\varphi\frac{\partial}{\partial y}}\right)$$

$$= i\hbar\left(-r\cos\Theta\frac{\partial}{\partial x} + r\sin\Theta\cos\varphi\frac{\partial}{\partial z}\right)$$

$$= i\hbar\left(-z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}\right) = z\hat{P}_x - x P_z$$

$$L_y = -i\hbar\frac{\partial}{\partial\varphi} = -i\hbar\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) = x\hat{P}_y - y\hat{P}_x$$