University of Calgary
Winter semester 2017

PHYS 543: Quantum Mechanics II

Homework assignment 1

Due September 27, 2017

Problem 1.1. The *single-photon added coherent state* (SPACS) is obtained from a coherent state by action of the creation operator:

$$|\alpha, 1\rangle = \mathcal{N}\hat{a}^\dagger |\alpha\rangle \,. \tag{1}$$

a) Find the normalization factor $\mathcal{N}$.

b) Find the decomposition of this state in the photon number basis.

c) Find the expectation value of the position observable.

d) Find the wavefunction of the SPACS for a real α by treating $a^\dagger = \frac{\hat{X}-i\hat{P}}{\sqrt{2}}$ in Eq. (1) as a differential operator.

e) Which quantum state does SPACS approach in the limit $\alpha = 0$? $\alpha \to \infty$?

f) Assuming α real, show that $|\alpha, 1\rangle = \hat{D}_X(\alpha\sqrt{2})(c\,|0\rangle + d\,|1\rangle)$ for certain values of c and d, where $\hat{D}_X(\cdot)$ is the position displacement operator. Find these values. Check that your result is consistent with part (c).
Hint: Use the relation $[\hat{a}^\dagger, \hat{D}_X(\alpha\sqrt{2})] = \alpha\hat{D}_X(\alpha\sqrt{2})]$ found in Homework 6 in the previous semester.

Problem 1.2. The operator $\hat{X}_\theta = \hat{X}\cos\theta + \hat{P}\sin\theta$ with a real θ is called the *quadrature observable*.

a) Write the commutation relation and uncertainty principle for the pair of quadrature observables $(\hat{X}_\theta, \hat{X}_{\frac{\pi}{2}+\theta})$.

b) Guess the single-oscillator Hamiltonian $\hat{H}$ for which the Heisenberg picture evolution is of the form

$$\hat{X}_\theta(t) = \hat{X}_\theta(0)e^{-\gamma t}; \tag{2}$$

$$\hat{X}_{\frac{\pi}{2}+\theta}(t) = \hat{X}_{\frac{\pi}{2}+\theta}(0)e^{\gamma t}. \tag{3}$$

Check your answer by writing the Heisenberg equations of motion for $\hat{X}_\theta$ and $\hat{X}_{\frac{\pi}{2}+\theta}$.
Hint: generalize the known answer for $\theta = 0$ to arbitrary values of θ.

c) Find the evolution of the state $|\psi(t)\rangle$ in the Schrödinger picture under this Hamiltonian with the initial state $|\psi(0)\rangle = |0\rangle$. Work in the Fock basis, decomposing the evolution operator into the Taylor series to the first order.
Hint: Check and use the identity $\hat{X}_\theta = \frac{1}{\sqrt{2}}(\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta})$.

d) Find the uncertainties of the observables $\hat{X}_\theta$ and $\hat{X}_{\frac{\pi}{2}+\theta}$ in the state $|\psi(t)\rangle$ and check that they are consistent with what is expected from the Heisenberg picture evolution.

e) Find the two-oscillator Hamiltonian $\hat{H}$ for which the Heisenberg picture evolution is of the form

$$\hat{X}_{\theta,\pm}(t) = \hat{X}_{\theta,\pm}(0)e^{\pm\gamma t}; \tag{4}$$

$$\hat{X}_{\frac{\pi}{2}+\theta,\pm}(t) = \hat{X}_{\frac{\pi}{2}+\theta,\pm}(0)e^{\mp\gamma t}, \tag{5}$$

where $\hat{X}_{\theta,\pm} = \frac{\hat{X}_{\theta,A}\pm\hat{X}_{\theta,B}}{\sqrt{2}}$; A and B label the two Hilbert spaces. Check your answer by writing the Heisenberg equations of motion for $\hat{X}_{\theta,\pm}$ and $\hat{X}_{\frac{\pi}{2}+\theta,\pm}$.

f) Find the evolution of the state $|\psi(t)\rangle$ in the Schrödinger picture under this Hamiltonian with the initial state $|\psi(0)\rangle = |0,0\rangle$. Work in the Fock basis, decomposing the evolution operator into the Taylor series to the first order.

g) Find the uncertainties of the observables $X_{\theta,\pm}$ and $X_{\frac{\pi}{2}+\theta,\pm}$ in the state $|\psi(t)\rangle$ and check that they are consistent with what is expected from the Heisenberg picture evolution.

Problem 1.3. Find the general form of the commutator $[\hat{L}_j, [\hat{L}_k, \hat{L}_l]]$ using the Levi-Civita formalism. Check your answer by specific examples: $[\hat{L}_x, [\hat{L}_y, \hat{L}_z]]$, $[\hat{L}_x, [\hat{L}_x, \hat{L}_z]]$ and $[\hat{L}_x, [\hat{L}_z, \hat{L}_x]]$. **Hint:** $\epsilon_{jkl}\epsilon_{jmn} = \delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}$.

Problem 1.4. From the expressions (4.25) for the angular momentum components in spherical coordinates in the lecture notes, derive these components in Cartesian coordinates [Eqs. (4.20)].