University of Calgary
Winter semester 2017

PHYS 543: Quantum Mechanics II

Final examination

December 13, 2017, 8:00–11:00, ST 127

Open books. No electronic equipment is allowed except basic calculators.
Full credit = 100 points. Partial credit will be given.

Problem 1. (15 pts) The motion of two particles is governed by the Hamiltonian

$$\hat{H} = \hbar\gamma(\hat{X}_A\hat{P}_B - \hat{P}_A\hat{X}_B),$$

where the rescaled position and momentum are used such that $[\hat{X}, \hat{P}] = i$.

a) (10 pts) Find the evolution $\hat{X}_{A,B}(t), \hat{P}_{A,B}(t)$ of the position and momentum operators as a function of time in the Heisenberg picture.

b) (2 pts) Can you think of a physical device that implements this evolution of the postition and momentum?

c) (3 pts) The state of the two particles in the Schrödinger picture is described by the wavefunction $\Psi(X_A, X_B, t)$. For which values of t is the condition

$$|\Psi(X_A, X_B, t)| = |\Psi(X_B, X_A, 0)|$$

fulfilled?

Problem 2. (10 pts) A particle is prepared in the state with the wavefunction $\psi(r, \theta, \phi) = R(r)Y(\theta, \phi)$, where

$$Y(\theta, \phi) = \frac{1}{\sqrt{8\pi}}\left(1 + \sqrt{3}\sin\theta\cos\phi\right).$$

Determine the possible outcomes of measurements of the operators $\hat{L}^2$ and $\hat{L}_z$ (where $\hat{\vec{L}}$ is the orbital angular momentum operator) as well as their respective probabilities.

Problem 3. (20 pts) An atom of tritium (^{3}H) in its ground state decays to singly-ionized helium (^{3}He$^+$) by beta decay, in a process which can be considered instantaneous on the atomic timescale. Calculate the probability that the electron in the helium ion is left in the ground state. The mass of the nucleus can be considered constant.
Hint: $\int\limits_0^\infty r^2 e^{-r/a}\mathrm{d}r = 2a^3$.

Problem 4. (20 pts) An electron in an atom is initially prepared in the state with the quantum numbers $l = 2, m_l = 1, m_s = \frac{1}{2}$. The electron evolves under the spin-orbit coupling Hamiltonian $\hat{H} = A\hat{\vec{L}}\cdot\hat{\vec{S}}$ for the time t. After that, the electron's spin is subjected to a Stern-Gerlach measurement with the magnetic field along the z axis. Find the splitting ratio that will be observed in this measurement.

Problem 5. (15 pts) The spin of electron 1 is in the state $\alpha|\uparrow\rangle + \beta|\downarrow\rangle$ while the spins of electrons 2 and 3 are in the coupled state with the quantum numbers corresponding to the total spin $\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$ equal to $s = 0$ and $m_s = 0$. A measurement of the projection of the total spin onto the z axis is performed on electrons 1 and 2 and reveals that this projection is also equal to zero. For the state of electron 3 after the measurement:

a) (10 pts) find the density operator;

b) (3 pts) find the length and direction of the Bloch vector;

c) (2 pts) find the state purity

as functions of α and β. No further information about the state of electrons 1 and 2 is available.

Problem 6. (20 pts) Alice and Bob share an optical two-mode squeezed vacuum state $|\Psi\rangle$ such that the position quadratures are correlated according to $\langle\Delta X_\pm^2\rangle = \frac{1}{2}e^{\pm 2r_s}$, where $X_\pm = \frac{X_A \pm X_B}{\sqrt{2}}$. Alice's mode propagates through an attenuator with the intensity transmissivity η. Find the new values of $\langle\Delta X_\pm^2\rangle$.