PHYS 543 Fall 2017

Final examination

Solutions

[1] a) $\dot{X}_A = \frac{i}{\hbar}[H, X_A] = -\frac{i\gamma}{\hbar}\hat{X}_B[P_A, X_A] = -\gamma X_B$

$\dot{X}_B = \gamma X_A$

$\dot{P}_A = \frac{i}{\hbar}[H, P_A] = \frac{i\gamma}{\hbar}[X_A, P_A]P_B = -\gamma P_B$

$\dot{P}_B = \gamma P_A$

$$\begin{cases} X_A(t) = X_A(0)\cos\gamma t - X_B(0)\sin\gamma t \\ X_B(t) = X_B(0)\cos\gamma t + X_A(0)\sin\gamma t \end{cases}$$

Same for momentum

b) Beam splitter

c) Full exchange for $\cos\gamma t = 0$, $|\sin\gamma t| = 1 \Rightarrow \gamma t = \frac{\pi}{2} + m\pi$

[2] $Y(\theta, \varphi) = \frac{1}{\sqrt{2}} Y_0^0(\theta, \varphi) - \frac{1}{2} Y_1^1(\theta\ \varphi) + \frac{1}{2} Y_1^{-1}(\theta\ \varphi)$

$pr(\ell = 0) = pr(\ell = 1) = 1/2$

$pr(m = 1) = pr(m = -1) = 1/2$

$pr(m = 0) = 1/2$

3 $\Psi_{100}(r\ \theta\ \varphi) = \frac{1}{\sqrt{\pi}} a^{-3/2} e^{-r/a}$

$a \propto \frac{1}{e^2} \Rightarrow a(^3H) = a_0 \qquad a(^3He^+) = \frac{1}{4} a_0$

$$pr = \left| \int \Psi_{100}(^3H)\, \Psi_{100}(^3He^+)\, d^3r \right|^2$$

$$= \left| \frac{1}{\pi}\left(\frac{1}{4}\right)^{-3/2} a_0^{-3} \int_0^\infty e^{-r/a_0} e^{-4r/a_0}\, 4\pi r^2\, dr \right|^2$$

$$= \left| 4 \cdot 8\, a_0^{-3} \int_0^\infty e^{-5r/a_0} r^2\, dr \right|^2$$

$$= \left| 4 \cdot 8 \cdot \frac{2}{5^3} \right|^2 = \left(\frac{64}{125}\right)^2 = 0.26$$

4 $|l=2, m_l=1, s=\frac{1}{2}, m_s=\frac{1}{2}\rangle = \sqrt{\frac{4}{5}}\, |j=\frac{5}{2}, m_j=\frac{3}{2}\rangle - \sqrt{\frac{1}{5}}\, |j=\frac{3}{2}, m_j=\frac{1}{2}\rangle$

$\hat{H} = A\, \vec{L}\cdot\vec{S} = \frac{A\hbar^2}{2}[j(j+1) - l(l+1) - s(s+1)]$

$j=\frac{5}{2} \to$ Energy $\frac{A\hbar^2}{2}\left[\frac{35}{4} - 6 - \frac{3}{4}\right] = A\hbar^2$

$j=\frac{3}{2} \to$ Energy $\frac{A\hbar^2}{2}\left[\frac{15}{4} - 6 - \frac{3}{4}\right] = -\frac{3A\hbar^2}{2}$

$\psi(t) = \sqrt{\frac{4}{5}}\, |j=\frac{5}{2}, m_j=\frac{3}{2}\rangle\, e^{-iA\hbar t} - \sqrt{\frac{1}{5}}\, |j=\frac{3}{2}, m_j=\frac{3}{2}\rangle\, e^{3iA\hbar t/2}$

$= \sqrt{\frac{4}{5}}\left(\sqrt{\frac{4}{5}}\, |m_l=1, m_s=\frac{1}{2}\rangle + \sqrt{\frac{1}{5}}\, |m_l=2, m_s=-\frac{1}{2}\rangle\right) e^{-iA\hbar t}$

$- \sqrt{\frac{1}{5}}\left(-\sqrt{\frac{1}{5}}\, |m_l=1, m_s=\frac{1}{2}\rangle + \sqrt{\frac{4}{5}}\, |m_l=2, m_s=-\frac{1}{2}\rangle\right) e^{3iA\hbar t/2}$

$= \left(\frac{4}{5} e^{-iA\hbar t} + \frac{1}{5} e^{3iA\hbar t/2}\right) |m_l=1, m_s=\frac{1}{2}\rangle$

$+ \frac{2}{5}\left(e^{-iA\hbar t} - e^{3iA\hbar t/2}\right) |m_l=2, m_s=-\frac{1}{2}\rangle\Big)$

$pr(m_s=\frac{1}{2}) = \left|\frac{4}{5} e^{-iA\hbar t} + \frac{1}{5} e^{3iA\hbar t/2}\right|^2$

$= \frac{17}{25} + \frac{8}{25}\cos\frac{A\hbar t}{2}$

$pr(m_s=-\frac{1}{2}) = \frac{4}{25}\left|e^{-iA\hbar t} - e^{3iA\hbar t/2}\right|^2$

$= \frac{8}{25} - \frac{8}{25}\cos\frac{A\hbar t}{2}$

$\boxed{5}$ a) $|S=0, m_s=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) = |\Psi^-\rangle$

Initial state: $(\alpha|\uparrow\rangle + \beta|\downarrow\rangle) \otimes |\Psi^-\rangle$

$$= \frac{\alpha}{\sqrt{2}}|\uparrow\uparrow\downarrow\rangle - \frac{\alpha}{\sqrt{2}}|\uparrow\downarrow\uparrow\rangle + \frac{\beta}{\sqrt{2}}|\downarrow\uparrow\downarrow\rangle - \frac{\beta}{\sqrt{2}}|\downarrow\downarrow\uparrow\rangle$$

Measurement yields $m_s = 0$ $\Rightarrow$

$\Rightarrow$ electrons 2 and 3 are either $|\uparrow\downarrow\rangle$ or $|\downarrow\uparrow\rangle$

In the first case, electron 1 is in state $\frac{\beta}{\sqrt{2}}|\downarrow\rangle$

In the second case $-\frac{\alpha}{\sqrt{2}}|\uparrow\rangle$

$$\rho = \frac{1}{2}|\alpha|^2|\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\beta|^2|\downarrow\rangle\langle\downarrow|$$

Factor $\frac{1}{2}$ drops due to renormalization

b) Bloch vector: $(0, 0, |\alpha|^2 - |\beta|^2)$, length $\left||\alpha|^2 - |\beta|^2\right|$

c) Purity $\mathrm{Tr}\,\rho^2 = |\alpha|^4 + |\beta|^4$

$\boxed{6}$ BS-model of absorption, Heisenberg picture:

$$X_A' = \sqrt{\eta}\, X_A + \sqrt{1-\eta}\ X_0$$

$$\langle \Delta X_\pm^2 \rangle = \frac{1}{2} \langle (X_A' \pm X_B)^2 \rangle$$

$$= \frac{1}{2} \langle (\sqrt{\eta}\, X_A + \sqrt{1-\eta}\ X_0 \pm X_B)^2 \rangle$$

$$= \frac{1}{2} \langle \left(\sqrt{\eta}\ \frac{X_+ + X_-}{\sqrt{2}} + \sqrt{1-\eta}\ X_0 \pm \frac{X_+ - X_-}{\sqrt{2}} \right)^2 \rangle$$

$$= \frac{1}{4} \langle \left[(\sqrt{\eta} \pm 1) X_+ + (\sqrt{\eta} \mp 1) X_- + \sqrt{2(1-\eta)}\ X_0 \right]^2 \rangle$$

$$= \frac{1}{4} \left[(\sqrt{\eta} + 1)^2 \langle X_+^2 \rangle + (1 \mp \eta) \langle X_-^2 \rangle + 2(1-\eta) \langle X_0^2 \rangle \right]$$

$$= \frac{1}{8} \left[(1 + \eta \pm 2\sqrt{\eta})\, e^{+2r_s} + (1 + \eta \mp 2\sqrt{\eta})\, e^{-2r_s} + 2(1-\eta) \right]$$

$$= \frac{1}{4} (1+\eta) \cosh 2r_s \pm \frac{1}{2} \sqrt{\eta} \sinh 2r_s + \frac{1}{4} (1-\eta)$$