University of Calgary
Fall semester 2015

PHYS 543: Quantum Mechanics II

Homework assignment 6

Due Tuesday, December 8, 2015

Problem 6.1. Calculate the Wigner functions of the states below. Calculate the probability densities $\mathrm{pr}(X)$ and $\tilde{\mathrm{pr}}(P)$ for the position and momentum from the states' wavefunctions or density matrices. Verify that these probability densities are obtained when the Wigner function is integrated over the other quadrature. For each case, plot the Wigner function and the two marginal distributions.

Do not use computers for calculations. Do use computers to generate graphics. Use $\alpha = 3$ for all plots.

a) vacuum state;

b) the single-photon state;

c) coherent state with a real, positive α;

d) coherent state with an imaginary α;

e) Schrödinger cat state $|\alpha\rangle - |-\alpha\rangle$ with a real α (neglect normalization).

f) state with the density matrix $(|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|)/2$.

Useful integrals:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ikx}e^{-x^2/4}\mathrm{d}x = \sqrt{2}e^{-k^2};$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} x^2 e^{ikx}e^{-x^2/4}\mathrm{d}x = -2\sqrt{2}e^{-k^2}\left(2k^2-1\right);$$

$$\int_{-\infty}^{+\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi}$$

$$\int_{-\infty}^{+\infty} x^2 e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

Problem 6.2. For the displacement operator $\hat{D}(X_0, 0)$:

a) Find $\hat{D}^\dagger(X_0,0)\hat{a}\hat{D}(X_0,0)$ and $\hat{D}^\dagger(X_0,0)\hat{a}^\dagger\hat{D}(X_0,0)$;

b) Find $[a, \hat{D}(X_0,0)]$ and $[a^\dagger, \hat{D}(X_0,0)]$;

c) Find the Fock decomposition of the *displaced single-photon state* $\hat{D}(X_0, 0)|1\rangle$ and the photon number distribution associated with this state.

Problem 6.3.

a) For the single-mode squeezed-vacuum state $\hat{S}(\zeta)|0\rangle$, calculate the uncertainties of the position and momentum observables using the Heisenberg picture. Verify the uncertainty principle.

b) For the two-mode squeezed-vacuum state $\hat{S}_2(\zeta)|00\rangle$, calculate the uncertainties of observables $\hat{X}_{1,2}$, $\hat{P}_{1,2}$, $\hat{X}_1 \pm X_2$ and $\hat{P}_1 \pm P_2$.

Problem 6.4. Consider a squeezed vacuum state with squeezing parameter ζ that has passed through an attenuator with intensity transmissivity η.

a) Using the beam splitter model of absorption in the Heisenberg picture, find the position and momentum uncertainties of the transmitted state.
Hint: Let the position and momentum observables of the initial squeezed state entering the fictitious beam splitter be $\hat{X}$ and $\hat{P}$ and those of the vacuum state $\hat{X}_v$ and $\hat{P}_v$. Express the position $\hat{X}'$ and momentum $\hat{P}'$ of the output mode through these observables

b) Express the initial state in the Fock basis to the first order of ζ. Then use the beam splitter model of absorption in the Schrödinger picture to find the density matrix of the transmitted state in the Fock basis. Determine the position and momentum uncertainties from that density matrix and verify consistency with part (a).

Problem 6.5. Channel A of a two-mode squeezed vacuum state with squeezing parameter $\zeta \ll 1$ is overlapped on a symmetric ($t = r = 1/\sqrt{2}$) beam splitter with a coherent state of amplitude $\alpha \ll 1$ (Fig. 1).

a) Find the decomposition of the resulting three-mode state in the Fock basis up to the first order in ζ and α.

b) One of the outputs of the beam splitter is measured with a photon number detector while the other one is discarded. Find the state of the channel B of the two-mode squeezed vacuum in the event the detector registers a single photon.

Problem 6.6. The Bloch vector of a spin adiabatically follows the magnetic field whose direction evolves as follows:

- along the meridian from the north pole to the position along the x axis;
- along the equator to the position along the y axis;
- along the meridian back to the north pole.

Find the geometric phase accumulated. Show that it is independent of the instantaneous velocity at each moment in time (as long as the adiabaticity criterion is fulfilled).

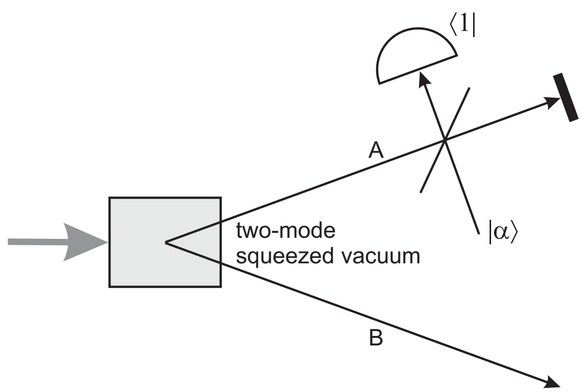

Figure 1: Illustration to Problem 6.4.