University of Calgary
Fall semester 2015

PHYS 543: Quantum Mechanics II

Homework assignment 5

Due Monday, November 16, 2015

Problem 5.1. Two electrons, whose spins are initially in state $|\Psi(0)\rangle = |+\rangle \otimes |\uparrow\rangle$, associated with fictitious observers Alice and Bob, are interacting with the Hamiltonian $\hat{H} = C\vec{S}_1 \cdot \vec{S}_2$.

a) Find the evolution $|\Psi(t)\rangle$ of the electrons' spin state in the canonical basis.

b) Alice measures the projection of her electron's spin onto the z axis at time t. Find the probabilities of possible results and the state in which Bob's electron will be prepared in each case. Based on that information, determine the ensemble describing the state of Bob's electron in case he does know the result of Alice's measurement. From that description, obtain the density matrix of Bob's electron in the canonical basis.

c) Repeat part (b) for Alice's measuring the projection of her electron's spin onto the x axis.

Parts below apply to Bob's electron only.

d) Find the density operator $\hat{\rho}_A(t)$ of Bob's electron as a function of time using the partial trace formalism. Verify that your result is identical to that of parts (b) and (c).

e) Find the Bloch vector trajectory and plot it.

f) Find the state purity as a function of time. Verify its relation with the length of the Bloch vector: $\mathrm{Tr}\hat{\rho}^2 = (|\vec{R}|^2 + 1)/2$.

g) At time $t = t_1$, the interaction between the two electrons is turned off, but a magnetic field $\vec{B}$ is turned on along the z axis. Find the density matrix as a function of time $t > t_1$ under these new conditions.

Problem 5.2. Consider states $\hat{\rho}_1 = |\psi_1\rangle\langle\psi_1| = \mathcal{N}_1(|\alpha\rangle + |-\alpha\rangle)(\langle\alpha| + \langle-\alpha|)$ and $\hat{\rho}_2 = \mathcal{N}_2(|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|)$ of a harmonic oscillator, where $|\pm\alpha\rangle$ are coherent states ($\alpha \in \mathbb{R}$) and $\mathcal{N}_{1,2}$ are normalization factors.

a) Find $\mathcal{N}_{1,2}$.
Hint: $\langle\alpha|\,\alpha'\rangle = e^{-|\alpha|^2/2-|\alpha'|^2/2+\alpha'\alpha^*}$.

b) Find the representation of the density operator of these states

- in the Fock basis;

- in the position basis;
- in the momentum basis.

Use software to make density plots of real parts of these density matrices/functions for $\alpha = 2$ in range $n \in [0, 10]$ in the Fock basis, $x, p \in [-5, 5]$ in the position and momentum bases.

Problem 5.3. An ensemble of electrons at zero temperature, initially in a pure state defined by Bloch vector $\vec{R}(0)$ with spherical coordinates $(1, \theta, 0)$, experiences decoherence and thermalization.

a) Find $\left.\frac{\mathrm{d}|R(t)|}{\mathrm{d}t}\right|_{t=0}$.

b) Find the upper limit $L(\theta)$ on T_2/T_1 that ensures that the above derivative is not positive (so the length of the Bloch vector do not exceed 1, and the state of the ensemble remains physically plausible during relaxation).

c) Find angle θ for which $L(\theta)$ is minimized. Find the corresponding value of $L(\theta)$.