University of Calgary
Fall semester 2015

PHYS 543: Quantum Mechanics II

Homework assignment 3

Due Wednesday, October 21, 2015, at 11:00 am

Problem 3.1.

a) A spin, detuned by $-\Delta$ from the rf field, initially in the $|\uparrow\rangle$ state, is subjected to an instant $\pi/2$ pulse at $t = 0$. Calculate the mean y component of the magnetic moment as a function of time $t > 0$.

b) An ensemble of spins is *inhomogeneously broadened*: each spin has a different Δ. The detunings follow the following distribution:

$$p(\Delta) = \frac{1}{\sqrt{\pi}\Delta_0} e^{-(\Delta/\Delta_0)^2}.$$

Averaging over the ensemble, under the conditions of part (a), calculate the mean y component of the magnetic moment as a function of time $t > 0$.

c) After time $t_0 \gg 1/\Delta_0$, the ensemble is hit by an instant π pulse. Calculate the mean y component of the magnetic moment as a function of time $t > t_0$.

d) Sketch the behavior of the collective magnetic component as a function of time, showing the behavior for all moments when it is significantly nonzero.

The phenomenon observed in part (c) is known as the *spin echo*.

Work in the rotating basis. Use the following assumptions when solving this problem.

- The Rabi frequency $\Omega \gg \Delta$, so the $\pi/2$ pulse can be considered precise in spite of the detunings.
- The fictitious magnetic field is oriented along the positive x axis.
- No damping or relaxation are present.

Problem 3.2. Show that, for any integer or semi-integer non-negative l_1, l_2, the state $|l_1, m_1 = l_1\rangle \otimes |l_2, m_2 = l_2\rangle$ is an eigenstate of operators $\hat{L}^2$ and $\hat{L}_z$ (where $\hat{\vec{L}} = \hat{\vec{L}}_1 + \hat{\vec{L}}_2$) with the eigenvalues corresponding to $l = l_1 + l_2, m = m_1 + m_2$.

Problem 3.3. Calculate all Clebsch-Gordan coefficients corresponding to the coupling between systems with $l_1 = \frac{3}{2}$ and $l_2 = \frac{1}{2}$, using the same approach as developed in the class.

- Find all possible values of l for which the Clebsch-Gordan coefficients do not vanish.
- Using the result of the previous problem, write

$$|l = 2, m = 2\rangle = |m_1 = 3/2, m_2 = 1/2\rangle \,. \tag{1}$$

  Obtain the corresponding Clebsch-Gordan coefficient.

- Apply the lowering operator $\hat{L}_- = \hat{L}_{1-} + \hat{L}_{2-}$ to both sides of Eq. (1) several times to find the decompositions of all states $|l = 2, m\rangle$ into the $|m_1, m_2\rangle$ basis. Express the coefficients of these decompositions as Clebsch-Gordan coefficients.

- States $|l=2, m=1\rangle$ and $|l=1, m=1\rangle$ both decompose into states $|m_1=3/2, m_2=-1/2\rangle$ and $|m_1=1/2, m_2=1/2\rangle$. Knowing the decomposition of state $|l=2, m=1\rangle$ from part (c), find the decomposition of $|l=1, m=1\rangle$ that is orthogonal. The overall phase of that decomposition is a matter of convention. Choose that phase so that the coefficient in front of $|m_1=3/2, m_2=-1/2\rangle$ is real and positive.
- Repeat the third step for $l=1$.

Problem 3.4. Alice and Bob each have a spin-1 particle. The added angular momentum of the two particles is in state $|l=2, m=-1\rangle$. Alice peforms a Stern-Gerlach measurement of her particle with the magnetic field along the x axis and detects the spin projection value $m_x=1$.

a) What state does this measurement prepare at Bob's location? The answer should be given in the eigenbasis of Bob's $\hat{L}_z$.

b) What is the probability of this event?

Problem 3.5. An electron ($s=1/2$) is in a state with orbital quantum number $l=1$. Initially at $t=0$, the electron is prepared in state $|m_l=0, m_s=1/2\rangle$. Spin-orbit interaction with Hamiltonian $\hat{H}=A\hat{\vec{L}}\cdot\hat{\vec{S}}$ is present.

a) Write the initial state in the $|j, m_j\rangle$ basis and find its evolution.

b) Write the state of the electron, as a function of time, in the $|m_l, m_s\rangle$ basis.

c) Find the probability to find the electron in a state with $m_s=1/2$ as a function of time.

Note: it is allowed to use the table of Clebsch-Gordan coefficients in Problems 3.4. and 3.5.