Phys 543 Fall 2015

Final examination

Solutions

$\boxed{1}$

a) $|j=1, m=-1\rangle = |\downarrow\downarrow\rangle$

$$pr_{|\uparrow\uparrow\rangle} = |\langle\uparrow\uparrow|\Psi\rangle|^2 = 9/11$$

b)
$$\rho = Tr_A |\Psi\rangle\langle\Psi| = {}_A\langle\uparrow|\Psi\rangle\langle\Psi|\uparrow\rangle_A + {}_A\langle\downarrow|\Psi\rangle\langle\Psi|\downarrow\rangle_A$$

$$= \frac{1}{11}\left(|\uparrow\rangle\langle\uparrow| + (i|\uparrow\rangle + 3|\downarrow\rangle)(-i\langle\uparrow| + 3\langle\downarrow|\right)$$

$$= \frac{1}{11}\left[\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}1 & 3i\\ -3i & 9\end{pmatrix}\right] = \frac{1}{11}\begin{pmatrix}2 & 3i\\ -3i & 9\end{pmatrix}$$

c)
$$\langle S_y\rangle = Tr(\rho S_y) = \frac{\hbar}{2}Tr(\rho\sigma_y)$$

$$= \frac{\hbar}{22}\, Tr\left[\begin{pmatrix}2 & 3i\\ -3i & 9\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right] =$$

$$= \frac{\hbar}{22}\left(-3 - 3\right) = -\frac{3\hbar}{11}$$

d)
$$pr_{\sigma_y=1} = \langle\sigma_y = 1|\rho|\sigma_y = 1\rangle$$

$$= \frac{1}{22}(1 \ \ -i)\begin{pmatrix}2 & 3i\\ -3i & 9\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{22}(1 \ \ -i)\begin{pmatrix}-1\\ 6i\end{pmatrix} = \frac{5}{22}$$

$$pr_{\sigma_y=-1} = \langle\sigma_y = -1|\rho|\sigma_y = -1\rangle$$

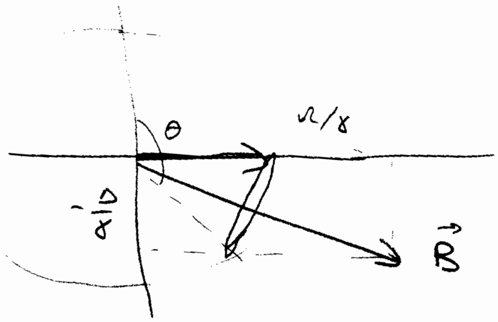
$$= \frac{1}{22}(1 \ \ i)\begin{pmatrix}2 & 3i\\ -3i & 9\end{pmatrix}\begin{pmatrix}1\\ -i\end{pmatrix} = \frac{1}{22}(1 \ \ i)\begin{pmatrix}5\\ -12i\end{pmatrix} = \frac{17}{22}$$

2 a) $H_{RWA} = \frac{\hbar}{2}\begin{pmatrix} \Delta & -\Omega \\ -\Omega & -\Delta \end{pmatrix}$ where $\Omega = \gamma B_{rf}/2$, $\Delta = \omega - \Omega_0$

b) $B_0 = \frac{\Omega_0}{\gamma} = \frac{\omega - \Delta}{\gamma}$

c) $\theta_B = -\arctan \Delta/\Omega$

d)



e) One-half Larmor period: $\frac{\pi}{\sqrt{\Omega^2 + \Delta^2}}$

f) $\theta = \frac{\pi}{2} + 2\arctan\frac{\Delta}{\Omega}$, $\varphi = 0$

g) $z = 1 - (1 - z_0)\, e^{-t/T_1}$, where $z_0 = -\tan\left(2\arctan\frac{\Delta}{\Omega}\right)$

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$$|j = \ell_1 + \ell_2, m = \ell_1 + \ell_2\rangle = |m_1 = \ell_1, m_2 = \ell_2\rangle$$

$$\Rightarrow \boxed{\langle \ell_1 + \ell_2, \ell_1 + \ell_2 | \ell_1, \ell_2\rangle = 1}$$

$$J_- |\ell_1 + \ell_2, \ell_1 + \ell_2\rangle = (L_{1-} + L_{2-})|\ell_1 \ \ell_2\rangle$$

$$\sqrt{(\ell_1+\ell_2)(\ell_1+\ell_2+1) - (\ell_1+\ell_2)(\ell_1+\ell_2-1)}\ |\ell_1+\ell_2, \ell_1+\ell_2-1\rangle$$

$$= \sqrt{\ell_1(\ell_1+1) - \ell_1(\ell_1-1)}\ |\ell_1-1 \ \ell_2\rangle + \sqrt{\ell_2(\ell_2+1) - \ell_2(\ell_2-1)}\ |\ell_1 \ \ell_2-1\rangle$$

$$\ell_2 = \tfrac{1}{2} \Rightarrow \sqrt{(\ell_1+\tfrac{1}{2})(\ell_1+\tfrac{3}{2}) - (\ell_1+\tfrac{1}{2})(\ell_1-\tfrac{1}{2})}\ |\ell_1+\tfrac{1}{2}, \ell_1-\tfrac{1}{2}\rangle$$

$$= \sqrt{2\ell_1}\ |\ell_1-1, \tfrac{1}{2}\rangle + |\ell_1, -\tfrac{1}{2}\rangle$$

$$\sqrt{2(\ell_1+\tfrac{1}{2})}\ |\ell_1+\tfrac{1}{2}, \ell_1-\tfrac{1}{2}\rangle = \sqrt{2\ell_1}\ |\ell_1-1, \tfrac{1}{2}\rangle + |\ell_1, -\tfrac{1}{2}\rangle \qquad (1)$$

$$\langle \ell_1+\tfrac{1}{2} \ \ell_1-\tfrac{1}{2} | \ell_1-1, \tfrac{1}{2}\rangle = \frac{\sqrt{2\ell_1}}{\sqrt{2\ell_1+1}}$$

$$\langle \ell_1+\tfrac{1}{2} \ \ell_1-\tfrac{1}{2} | \ell_1, -\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2\ell_1+1}}$$

State orthogonal to (1):

$$|\ell_1-\tfrac{1}{2}, \ell_1-\tfrac{1}{2}\rangle = \pm \frac{1}{\sqrt{2\ell_1+1}}\ |\ell_1-1, \tfrac{1}{2}\rangle \mp \frac{\sqrt{2\ell_1}}{\sqrt{2\ell_1+1}}\ |\ell_1, -\tfrac{1}{2}\rangle$$

$$\langle \ell_1-\tfrac{1}{2}, \ell_1-\tfrac{1}{2} | \ell_1-1, \tfrac{1}{2}\rangle = \pm \frac{1}{\sqrt{2\ell_1+1}}$$

$$\langle \ell_1-\tfrac{1}{2}, \ell_1-\tfrac{1}{2} | \ell_1, -\tfrac{1}{2}\rangle = \mp \frac{\sqrt{2\ell_1}}{\sqrt{2\ell_1+1}}$$

[4]

a) $H = \hbar A X_1 P_2$

$$\dot{X}_1 = \frac{i}{\hbar}[H\ X_1] = 0$$

$$\dot{X}_2 = \frac{i}{\hbar}[H\ X_2] = \frac{i}{\hbar}(-i)\hbar A X_1 = A X_1$$

$$\dot{P}_1 = \frac{i}{\hbar}[H\ P_1] = \frac{i}{\hbar}(i)\hbar A P_2 = -A P_2$$

$$\dot{P}_2 = \frac{i}{\hbar}[H\ P_2] = 0$$

b)
$$X_1(t) = X_1(0)$$

$$X_2(t) = X_2(0) + A X_1(0) t$$

$$P_1(t) = P_1(0) - A P_2(0) t$$

$$P_2(t) = P_2(0)$$

c)
$$\langle X_{1,2}(t)\rangle = \langle P_{1,2}(t)\rangle = 0$$

$$\langle \Delta (X_1 \pm X_2)^2 \rangle = \langle \Delta [(1 \pm At) X_1(0) + X_2(0)]^2 \rangle$$

$$= \frac{1}{2}\left[(1 \pm At)^2 + 1\right] \qquad (= 1 \text{ for } t = 0)$$

$$\langle \Delta (P_1 \pm P_2)^2 \rangle = \langle \Delta [P_1(0) \pm (-At \pm 1) P_2(0)]^2 \rangle$$

$$= \frac{1}{2}\left[1 + (-At \pm 1)^2\right] \qquad (= 1 \text{ for } t = 0)$$

Squeezing is present if either $(1 \pm At)^2$ or $(-At \pm 1)^2$ are less than one, i.e. when $At < 2$.

d)
$$e^{-iHt/\hbar}|00\rangle$$

$$= \left(1 - i\frac{Ht}{\hbar}\right)|00\rangle$$

$$= (1 - i A X_1 P_2 t)|00\rangle$$

$$= |00\rangle - iAt\,\frac{(a + a^+)(a - a^+)}{2}|00\rangle$$

$$= |00\rangle + \frac{At}{2}|11\rangle \equiv |\Psi\rangle$$

e)
$$\langle \Psi | (X_1 \pm X_2)^2 | \Psi \rangle$$

$$= \left(\langle 00| + \frac{At}{2}\langle 11|\right)\left(\frac{a_1 + a_1^+ \pm a_2 \pm a_2^+}{\sqrt{2}}\right)^2\left(|00\rangle + \frac{At}{2}|11\rangle\right)$$

$$= \frac{1}{2}\Big[\langle 00| a_1 a_1^+ + a_2 a_2^+ |00\rangle \pm \frac{At}{2}\langle 00|a_1 a_2|11\rangle \pm \frac{At}{2}\langle 11|a_1^+ a_2^+|00\rangle$$

$$= 1 \pm At$$

Same as $\frac{1}{2}[(1 \pm At)^2 + 1]$ for $At \ll 1$.

$\boxed{5}$

a) $S_z = \frac{\hbar}{2}\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right)$

$$H = C\hbar\, S_z \otimes S_z = \frac{C\hbar^3}{4}\left(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| - |\uparrow\downarrow\rangle\langle\uparrow\downarrow| - |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|\right)$$

$$= \frac{C\hbar^3}{4}\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&1\end{pmatrix}$$

b) $|\psi(0)\rangle = \frac{1}{2}\left(|\uparrow\rangle + |\downarrow\rangle\right) \otimes \left(|\uparrow\rangle + |\downarrow\rangle\right) = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

c) $$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle = \frac{1}{2}\begin{pmatrix}e^{-i\omega t}&&&\\&e^{i\omega t}&&\\&&e^{i\omega t}&\\&&&e^{-i\omega t}\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}e^{-i\omega t}\\e^{i\omega t}\\e^{i\omega t}\\e^{-i\omega t}\end{pmatrix}$$

where $\omega = \frac{C\hbar^2}{4}$

d) $|\psi(t)\rangle = \frac{1}{2}\left(e^{-i\omega t}|\uparrow\uparrow\rangle + e^{i\omega t}|\uparrow\downarrow\rangle + e^{i\omega t}|\downarrow\uparrow\rangle + e^{-i\omega t}|\downarrow\downarrow\rangle\right)$

${}_A\langle\uparrow|\psi(t)\rangle = \frac{1}{2}\left(e^{-i\omega t}|\uparrow\rangle + e^{i\omega t}|\downarrow\rangle\right) = \frac{1}{2}\begin{pmatrix}e^{-i\omega t}\\e^{i\omega t}\end{pmatrix}$ $\quad$ pr $= \frac{1}{2}$

${}_A\langle\downarrow|\psi(t)\rangle = \frac{1}{2}\left(e^{i\omega t}|\uparrow\rangle + e^{-i\omega t}|\downarrow\rangle\right) = \frac{1}{2}\begin{pmatrix}e^{i\omega t}\\e^{-i\omega t}\end{pmatrix}$ $\quad$ pr $= \frac{1}{2}$

$$\rho = \frac{1}{4}\begin{pmatrix}1 & e^{-2i\omega t}\\ e^{2i\omega t} & 1\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1 & e^{2i\omega t}\\ e^{2i\omega t} & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & \cos 2\omega t\\ \cos 2\omega t & 1\end{pmatrix}$$

e) ${}_A\langle +|\psi(t)\rangle = \frac{1}{2\sqrt{2}}\left(e^{-i\omega t}|\uparrow\rangle + e^{i\omega t}|\downarrow\rangle + e^{i\omega t}|\uparrow\rangle + e^{-i\omega t}|\downarrow\rangle\right)$

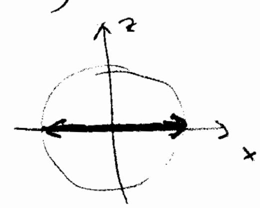
$= \frac{1}{\sqrt{2}}\cos\omega t\left(|\uparrow\rangle + |\downarrow\rangle\right) = \cos\omega t\,|+\rangle$ $\quad$ pr $= \cos^2\omega t$

${}_A\langle -|\psi(t)\rangle = \frac{1}{2\sqrt{2}}\left(e^{-i\omega t}|\uparrow\rangle + e^{i\omega t}|\downarrow\rangle - e^{i\omega t}|\uparrow\rangle - e^{-i\omega t}|\downarrow\rangle\right)$

$= \frac{i}{\sqrt{2}}\sin\omega t\left(|\downarrow\rangle - |\uparrow\rangle\right) = -i\sin\omega t\,|-\rangle$ $\quad$ pr $= \sin^2\omega t$

$$\rho = \cos^2\omega t\,|+\rangle\langle +| + \sin^2\omega t\,|-\rangle\langle -| = \frac{1}{2}\begin{pmatrix}1 & \cos 2\omega t\\ \cos 2\omega t & 1\end{pmatrix}$$

f) $\vec{R} = \frac{1}{2}\left(\cos^2\omega t\; R_{|+\rangle} + \sin^2\omega t\; R_{|-\rangle}\right)$



g) $H' = \mu B = \mu_B B\,\sigma_z = \mu_B B\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$$e^{-iH'(t-t_1)/\hbar} = \begin{pmatrix}e^{-i\Omega(t-t_1)} & 0\\ 0 & e^{i\Omega(t-t_1)}\end{pmatrix} \quad \text{where} \quad \Omega = \frac{\mu_B B}{\hbar}$$

$$\rho(t) = e^{-iH'(t-t_1)/\hbar}\rho(t_1)e^{iH'(t-t_1)/\hbar} = \frac{1}{2}\begin{pmatrix}1 & \cos 2\omega t_1\, e^{-2i\Omega(t-t_1)}\\ \cos 2\omega t_1\, e^{2i\Omega(t-t_1)} & 1\end{pmatrix}$$