University of Calgary
Fall semester 2015

PHYS 543: Quantum Mechanics II

Final examination

December 12, 2015, 19:00–22:00

Open books. No electronic equipment allowed.
Problems 1, 2 and 3 are mandatory. Choose **one** of Problems 4 and 5.
Full credit = 100 points. Partial credit will be given.

Problem 1 (20 points). Alice's and Bob's electrons are initially in state

$$|\Psi\rangle = \frac{1}{\sqrt{11}}(|\uparrow\uparrow\rangle + i\,|\downarrow\uparrow\rangle + 3\,|\downarrow\downarrow\rangle).$$

a) What is the probability to find the electron pair in state $|j = 1, m = -1\rangle$?

b) Alice's electron is lost. Find the density matrix of Bob's electron in the canonical basis.

c) Find the expectation value of operator $\hat{S}_y$ for Bob's electron.

d) Bob performs a Stern-Gerlach experiment on his electron, with the magnetic field oriented along the y axis. Find the probability of each outcome.

Problem 2 (20 points). A nuclear magnetic resonance experiment is performed with a proton whose spin is initially oriented along the x axis. The frequency ω of the rf field of amplitude $B_{\rm rf}$ is slightly higher than the resonance frequency Ω_0.

a) Write the rotating-wave Hamiltonian, expressing all of its terms through ω, Ω_0, $B_{\rm rf}$ and the gyromagnetic ratio γ.

b) Express the dc field B_0 through the same parameters.

c) Assuming that the fictitious magnetic field vector is in the x–z plane, find the angle between that vector and the x axis.

d) Sketch the trajectory of the Bloch vector.

e) How long will it take the Bloch vector to reach the most distant point of its trajectory for the first time?

f) Find the spherical and Cartesian coordinates of the Bloch vector at that point.

g) When the Bloch vector reaches the most distant point of its trajectory, the rf field is turned off. Find the z component of the Bloch vector as a function of time given the transverse and longitudinal relaxation time constants, T_1 and T_2 respectively. Assume absolute zero temperature.

Problem 3 (20 points). For the coupling of two particles with angular momenta l_1 and l_2, with l_1 being arbitrary and $l_2 = \frac{1}{2}$, find the Clebsch-Gordan coefficients $\langle j, m|\, m_1, m_2\rangle$ for

a) $m_1 = l_1,\ m_2 = \frac{1}{2},\ j = l_1 + \frac{1}{2},\ m_j = l_1 + \frac{1}{2};$

b) $m_1 = l_1,\ m_2 = -\frac{1}{2},\ j = l_1 + \frac{1}{2},\ m_j = l_1 - \frac{1}{2};$

c) $m_1 = l_1 - 1,\ m_2 = \frac{1}{2},\ j = l_1 + \frac{1}{2},\ m_j = l_1 - \frac{1}{2}$;

d) $m_1 = l_1,\ m_2 = -\frac{1}{2},\ j = l_1 - \frac{1}{2},\ m_j = l_1 - \frac{1}{2}$;

e) $m_1 = l_1 - 1,\ m_2 = \frac{1}{2},\ j = l_1 - \frac{1}{2},\ m_j = l_1 - \frac{1}{2}$.

An arbitrary phase convention can be adopted, but it must be consistent throughout your solution.

Problem 4 (40 points). Two optical modes (or two harmonic oscillators), initially in the vacuum state $|0\rangle \otimes |0\rangle$, interact under Hamiltonian

$$\hat{H} = \hbar A \hat{X}_1 \hat{P}_2$$

with a real and positive A. The position and momentum observables are rescaled, i.e. $[\hat{X}, \hat{P}] = i$.

a) Find the expectation values $\langle X_{1,2}\rangle, \langle P_{1,2}\rangle$ and mean square uncertainties $\left\langle \Delta(X_1 \pm X_2)^2 \right\rangle, \left\langle \Delta(P_1 \pm P_2)^2 \right\rangle$ at $t = 0$.

b) Write the differential equations for the position and momentum observables $\hat{X}_{1,2}(t)$ and $\hat{P}_{1,2}(t)$ in the Heisenberg picture.

c) Solve these equations and obtain the expressions for $\hat{X}_{1,2}(t)$ and $\hat{P}_{1,2}(t)$.

d) Find the expectation values $\langle X_{1,2}\rangle, \langle P_{1,2}\rangle$ and mean square uncertainties $\left\langle \Delta(X_1 \pm X_2)^2 \right\rangle, \left\langle \Delta(P_1 \pm P_2)^2 \right\rangle$ as functions of time t.

e) For which values of t is two-mode squeezing present, i.e. one of the above uncertainties is below that of the vacuum state at time $t = 0$?

f) Find the first-order approximation of the state into which the double-vacuum state evolves under Hamiltonian $\hat{H}$ in the Fock basis, in the Schrödinger picture, assuming $At/\hbar \ll 1$.

g) Find the mean square value $\left\langle (X_1 \pm X_2)^2 \right\rangle$ of that state. Is your result consistent with that of part (d)?

Problem 5 (40 points). Two electrons, whose spins are initially in state $|\Psi(0)\rangle = |+\rangle \otimes |+\rangle$, associated with fictitious observers Alice and Bob, are interacting with the Hamiltonian

$$\hat{H} = \hbar C \hat{S}_z \otimes \hat{S}_z.$$

a) Write the Hamiltonian in the canonical basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ in the Dirac and matrix notations.

b) Write the initial state in the canonical basis in the Dirac and matrix notations.

c) Find the evolution $|\Psi(t)\rangle$ of the electrons' spin state in the canonical basis.

d) Alice measures the projection of her electron's spin onto the z axis at time t. Find the probabilities of possible results and the state in which Bob's electron will be prepared in each case. Based on that information, determine the ensemble describing the state of Bob's electron in case he does know the result of Alice's measurement. From that description, obtain the density matrix $\hat{\rho}(t)$ of Bob's electron in the canonical basis.

e) Repeat part (d) for Alice's measuring the projection of her electron's spin onto the x axis.

f) Find the trajectory of the Bloch vector of Bob's electron and plot it.

g) At time $t = t_1$, the interaction between the two electrons is turned off, but a magnetic field $\vec{B}$ is turned on along the z axis. Find the density matrix of Bob's electron as a function of time $t > t_1$ under these new conditions.