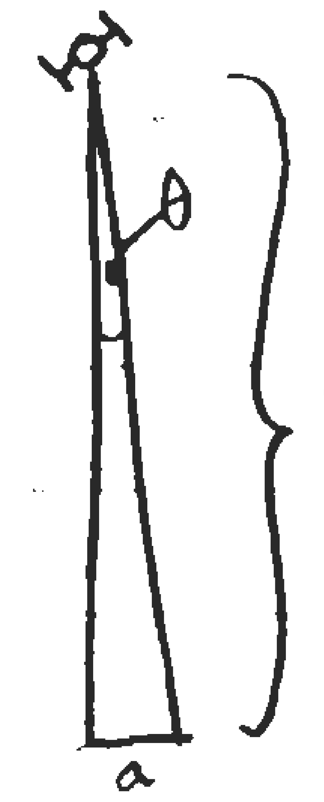
1. 

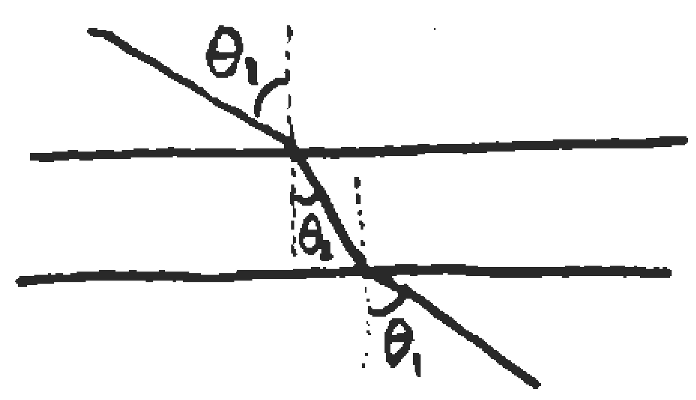
$\tan\theta = a/h \approx \theta$

The angular resolution of a camera is $\Delta\theta = 1.22\,\lambda/D$

So $D \approx 1.22\lambda\, h/a = \dfrac{1.22\,(550\times10^{-9}m)(3\times10^{5}m)}{0.30m} = 0.67m$

So the size of the lens is roughy 70cm

2. We will use the reference frame where $H \to E_{\parallel}$ in the Fresnel equations and $V \to E_{\perp}$ in the Fresnel equations. Our intial polarization vector is: $|R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle) \Leftrightarrow \frac{1}{\sqrt{2}}(\hat{e}_{\parallel} + i\hat{e}_{\perp})$



$\dfrac{\sin\theta_1}{\sin\theta_2} = 1.5 \qquad \theta_2 = \sin^{-1}\left(\dfrac{2}{3\sqrt{2}}\right) = 28.1^\circ = 0.491\,rad$

The total transmission for each component is the product of the transmissions at each interface: $t_{tot\parallel} = t_{1\parallel}\, t_{2\parallel} \qquad t_{tot\perp} = t_{1\perp}\, t_{2\perp}$

$$t_{tot\parallel} = \left(\frac{2n_1\cos\theta_1}{n_1\cos\theta_2 + n_2\cos\theta_1}\right)\left(\frac{2n_2\cos\theta_2}{n_2\cos\theta_1 + n_1\cos\theta_2}\right) = \frac{4\cdot1.5\cdot\frac{1}{\sqrt{2}}\cdot\cos(0.491)}{\left(\cos(0.491) + \frac{1.5}{\sqrt{2}}\right)\left(\frac{1.5}{\sqrt{2}} + \cos(0.491)\right)}$$

$= 0.99$

$$t_{tot\perp} = \left(\frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2}\right)\left(\frac{2n_2\cos\theta_2}{n_2\cos\theta_2 + n_1\cos\theta_1}\right) = \frac{4\cdot1.5\cdot\frac{1}{\sqrt{2}}\cos(0.491)}{\left(\frac{1}{\sqrt{2}}\cdot1.5\cos(0.491)\right)^2} = 0.91$$

So $\vec{P}_{final} = \dfrac{1}{N}\begin{pmatrix}0.99\\ i0.91\end{pmatrix} \qquad N = \sqrt{(0.99)^2 + (0.91)^2} = 1.34$

The final polarization is $\boxed{\dfrac{1}{1.34}\begin{pmatrix}0.99\\ i0.91\end{pmatrix}}$

3. $|R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ $|L\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

We represent the rotation of the coordinate system in a matrix:

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$R|R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta + i\sin\theta \\ -\sin\theta + i\cos\theta \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\theta} \\ ie^{i\theta} \end{pmatrix} = e^{i\theta}|R\rangle$$

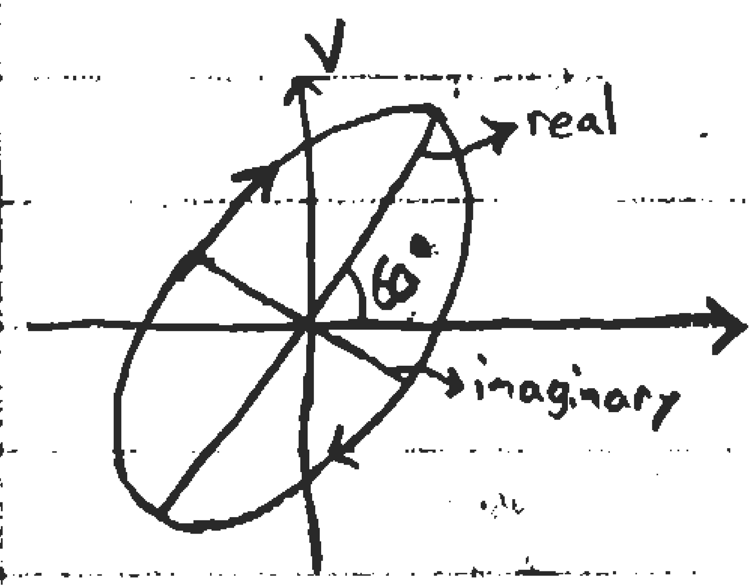
So the overall phase of $|R\rangle$ changes but the light is always in the state $|R\rangle$

Similarity: $$R|L\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta - i\sin\theta \\ -\sin\theta - i\cos\theta \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\theta} \\ -ie^{-i\theta} \end{pmatrix} = e^{-i\theta}|L\rangle$$

4.a) For maximum transmission the axis will lie along the major axis of the ellipse; at 60°. Minimum transmission will occur at an orthgonal angle; at 150° (-30°).

b) We can choose to intially work in the basis defined by the major axis or start directly in the final basis.

Let's choose the major axis component to be purely real, hence the minor axis will be imaginary.



$$E_{ox} = E_{o_{Maj}} \cos(60°) + i E_{o_{min}} \sin(60°) = \sqrt{I_{max}}\,\tfrac{1}{2} + i\sqrt{I_{min}}\,\tfrac{\sqrt{3}}{2}$$
$$E_{oy} = E_{o_{Maj}} \sin(60°) - i E_{o_{min}} \cos(60°) = \sqrt{I_{max}}\,\tfrac{\sqrt{3}}{2} - i\sqrt{I_{min}}\,\tfrac{1}{2}$$

$$|P\rangle = N\begin{pmatrix} \sqrt{20} + i\sqrt{15} \\ \sqrt{60} - i\sqrt{5} \end{pmatrix} = \frac{1}{2\sqrt{5}}\begin{pmatrix} 2 + i\sqrt{3} \\ 2\sqrt{3} - i \end{pmatrix}$$

Or, we can start in the basis defined by the axes of the ellipse:

$$|P_o\rangle = N\begin{pmatrix} \sqrt{I_{max}} \\ -i\sqrt{I_{min}} \end{pmatrix} \qquad |P\rangle = R\,|P_o\rangle$$

$$|P\rangle = N\begin{pmatrix} \cos(60°) & -\sin(60°) \\ \sin(60°) & \cos(60°) \end{pmatrix}\begin{pmatrix} \sqrt{I_{max}} \\ -i\sqrt{I_{min}} \end{pmatrix} = N\begin{pmatrix} \frac{1}{2}\sqrt{20} + i\frac{1}{2}\sqrt{15} \\ \frac{1}{2}\sqrt{60} - i\frac{1}{2}\sqrt{5} \end{pmatrix}$$

$$|P\rangle = \frac{1}{2\sqrt{5}}\begin{pmatrix} 2 + i\sqrt{3} \\ 2\sqrt{3} - i \end{pmatrix}$$

c) $M_\phi = R^{-1} M_o R$ where $M_o = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $R = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$

$$R^{-1} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \qquad M_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos^2\phi & -\sin\phi\cos\phi \\ -\sin\phi\cos\phi & \sin^2\phi \end{pmatrix}$$

$$M_\phi = \begin{pmatrix} \cos^2\phi & -\sin\phi\cos\phi \\ -\sin\phi\cos\phi & \sin^2\phi \end{pmatrix}$$

d) Let's work in the basis defined by the major axis of the ellipse: $|P\rangle = \begin{pmatrix} \sqrt{20\,W/m^2} \\ -i\sqrt{5\,W/m^2} \end{pmatrix}$ (dropping normalization because we want to track intensity)

$$M_\phi |P\rangle = \begin{pmatrix} \cos^2\phi & -\sin\phi\cos\phi \\ -\sin\phi\cos\phi & \sin^2\phi \end{pmatrix} \begin{pmatrix} \sqrt{20\,W/m^2} \\ -i\sqrt{5\,W/m^2} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{I_{max}}\cos^2\phi + i\sqrt{I_{min}}\sin\phi\cos\phi \\ -\sqrt{I_{max}}\sin\phi\cos\phi - i\sqrt{I_{min}}\sin^2\phi \end{pmatrix}$$

Keep in mind that this is with ~~respect to the~~ major axis

$$I(\phi) = |E_{ox}|^2 + |E_{oy}|^2 = I_{max}\cos^4\phi + I_{min}\cos^2\phi\sin^2\phi + 2\sqrt{I_{max}I_{min}}\cos^2\phi\sin\phi\cos\phi$$
$$+ I_{max}\sin^2\phi\cos^2\phi + I_{min}\sin^4\phi + 2\sqrt{I_{max}I_{min}}\sin^2\phi\sin\phi\cos\phi$$

$$I(\phi - 60^\circ) = I(\theta) = I_{max}\cos^2\theta + I_{min}\sin^2\theta + 2\sqrt{I_{max}I_{min}}\sin\theta\cos\theta$$

$$I(\phi - 60^\circ) = I(\theta) = \tfrac{1}{2}\left(I_{max} + I_{min} + \cos(2\theta)(I_{max} - I_{min})\right) + \sin(2\theta)\sqrt{I_{max}I_{min}}$$

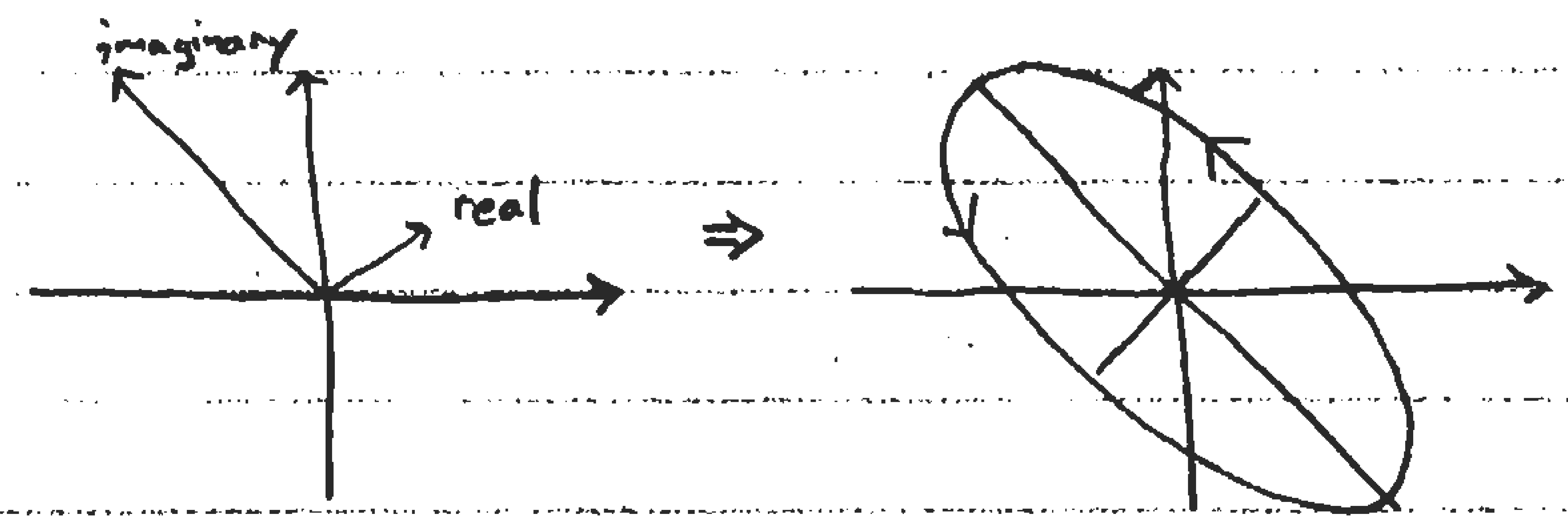
At $\phi = 45^\circ$, $\theta = -15^\circ$ $\quad I(\phi = 45^\circ) = \frac{1}{2}\left[25\,W/m^2 + \frac{\sqrt{3}}{2}(15\,W/m^2)\right] + \frac{1}{2}\sqrt{100\,W^2/m^4}$

$$I(\phi = 45^\circ) = 14\,W/m^2$$

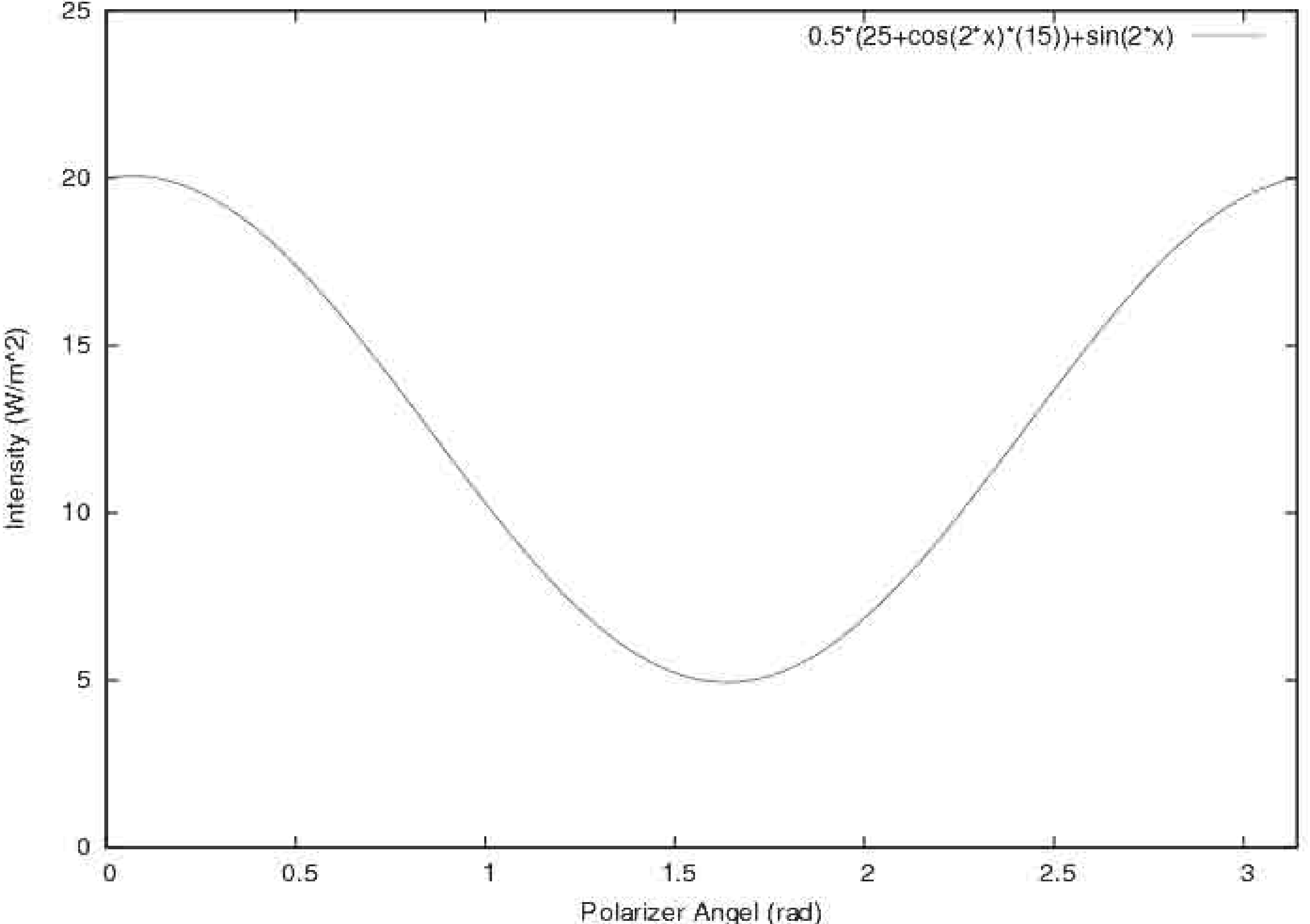
e) $M_{\lambda/2_0} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ $\quad M_{\lambda/2_0}|P_0\rangle = \frac{1}{2\sqrt{5}}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 2 + i\sqrt{3} \\ 2\sqrt{3} - i \end{pmatrix} = \frac{1}{2\sqrt{5}}\begin{pmatrix} -2i + \sqrt{3} \\ 2\sqrt{3}i + 1 \end{pmatrix}$

We can place the real and imaginary vectors in cartesian coordinates to visualize the polarization:

$(\sqrt{3}, 1)$
$i(-2, 2\sqrt{3})$



The handedness has changed.


0.5*(25+cos(2*x)*(15))+sin(2*x)
Intensity (W/m^2)
Polarizer Angel (rad)
25
20
15
10
5
0
0
0.5
1
1.5
2
2.5
3

5. a) HWP at an angle θ:

$$M_{\lambda/2\theta} = R^{-1} M_{\lambda/2_0} R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -i\cos\theta & i\sin\theta \\ i\sin\theta & i\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} -i\cos^2\theta + i\sin^2\theta & 2i\sin\theta\cos\theta \\ 2i\sin\theta\cos\theta & -i\sin^2\theta + i\cos^2\theta \end{pmatrix}$$

$$M_{\lambda/2}(\theta) = i\begin{pmatrix} -\cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix}$$

QWP with the fast axis horizontal:

$$M_{\lambda/4_0} = e^{i\pi/4}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

b) After the first wave-plate:

$$|P_1\rangle = M_{\lambda/2\theta}|P_0\rangle = \frac{1}{2\sqrt{5}}\begin{pmatrix} -i\cos(2\theta) & i\sin(2\theta) \\ i\sin(2\theta) & i\cos(2\theta) \end{pmatrix}\begin{pmatrix} 2+i\sqrt{3} \\ 2\sqrt{3}-i \end{pmatrix}$$

for $\theta/2 = 30^\circ$: $$M_{\lambda/2}(30^\circ)|P_0\rangle = \frac{1}{2\sqrt{5}}\begin{pmatrix} -i\,1/2 & i\sqrt{3}/2 \\ i\sqrt{3}/2 & i\,1/2 \end{pmatrix}\begin{pmatrix} 2+i\sqrt{3} \\ 2\sqrt{3}-i \end{pmatrix}$$

$$= \frac{1}{2\sqrt{5}}\begin{pmatrix} -i + \sqrt{3}/2 + i3 + \sqrt{3}/2 \\ i\sqrt{3} - 3/2 + i\sqrt{3} + 1/2 \end{pmatrix}$$

$$|P_1\rangle = \frac{1}{2\sqrt{5}}\begin{pmatrix} \sqrt{3} + 2i \\ -1 + \sqrt{3}/2\,i \end{pmatrix}$$ After the first wave-plate

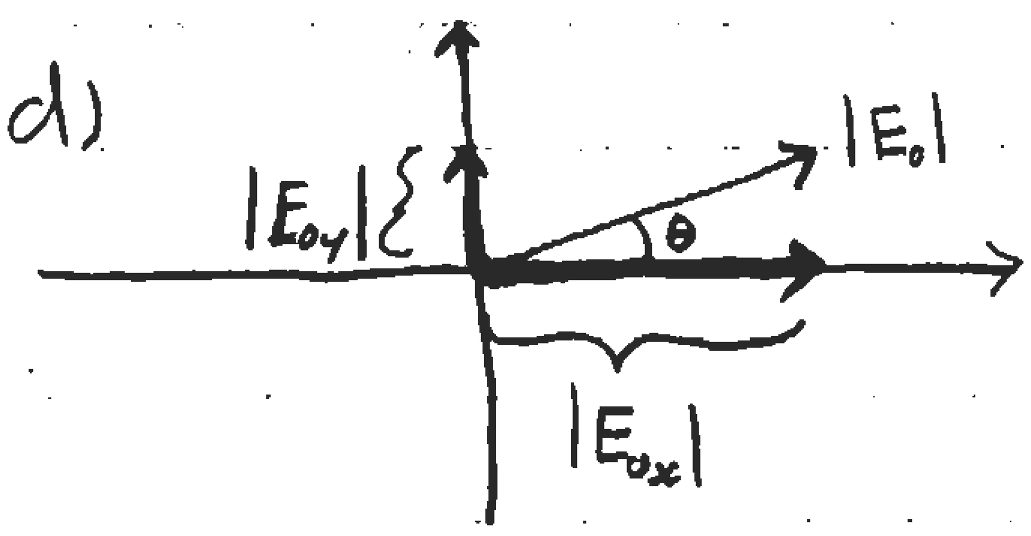
After the second wave-plate:

$$|P_2\rangle = \frac{e^{i\pi/4}}{2\sqrt{5}}\begin{pmatrix}\sqrt{3}+2i\\ -\sqrt{3}/2-i\end{pmatrix}$$

c) $$\frac{e^{i\pi/4}}{2\sqrt{5}}\begin{pmatrix}\sqrt{3}+2i\\ \sqrt{3}/2-i\end{pmatrix} = \frac{e^{i\pi/4}}{2\sqrt{5}}\begin{pmatrix}\sqrt{7}\,e^{i\phi}\\ -\sqrt{7}/2\,e^{i\phi}\end{pmatrix} = \frac{e^{i(\pi/4+\phi)}}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix}$$ where $\phi = \tan^{-1}\left(\frac{Im}{Re}\right)$

This makes sense because the $\lambda/2$-plate rotates (more correctly it reflects rather than rotates) the polarization so that the major and minor axes lie along x and y. The $\lambda/4$-plate then retards the y component so that it is in phase with the x component.

d)



$$\theta = \tan^{-1}\left(\frac{|E_{oy}|}{|E_{ox}|}\right) = \tan^{-1}\left(\frac{\sqrt{3/4+1}}{\sqrt{3+4}}\right)$$

$$\theta = \tan^{-1}\left(\sqrt{1/4}\right) = \tan^{-1}(1/2)$$

$$\theta = 26.6^\circ$$

$$|E_o| \propto \sqrt{|E_{omaj}|^2 + |E_{omin}|^2} = \sqrt{I_{omaj}+I_{omin}}$$

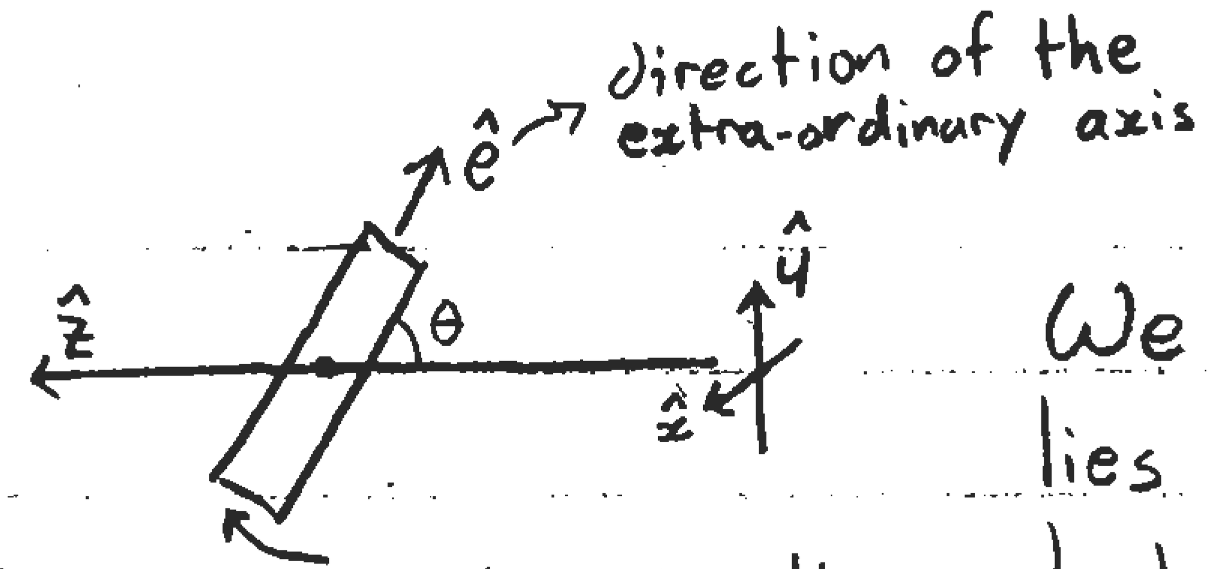
$$I \propto |E_o|^2 = I_{omaj} + I_{omin} = 20\,W/m^2 + 5\,W/m^2$$

$$I = 25\,W/m^2$$

e) A half waveplate will reflect a polarization about its axis, so:

$\theta'_{\lambda/2} = \frac{1}{2}(90^\circ - 26.6^\circ) = 31.7^\circ$ with respect to the vertical axis

$$\theta_{\lambda/2} = 90^\circ - 31.7^\circ = 58.3^\circ$$

6. 

We rotate the crystal in a way that E_x lies along an ordinary axis for all angles but E_y varies from an extra-ordinary axis to an ordinary one.

★ $\hat{x}$ is aligned with the ordinary axis.

$$n_e(\theta) = n_o \sqrt{\frac{1+\tan^2\theta}{1+(n_o/n_e)^2\tan^2\theta}}$$

First, try phase-matching with the pump polarized along the ordinary axis.

$$n_o(\lambda_1) = n_{e_{\lambda_2}}(\theta) = n_o(\lambda_2)\sqrt{\frac{1+\tan^2\theta}{1+(n_o(\lambda_2)/n_e(\lambda_2))^2\tan^2\theta}}$$

$$1.656 = 1.678\sqrt{\frac{1+\tan^2\theta}{1+\left(\frac{1.678}{1.557}\right)^2\tan^2\theta}} \Rightarrow \theta = 24^\circ$$

So, with the pump along the x axis (as shown in the diagram) phase matching is achieved at 24°.

7. $x(t) = \frac{q_e/m_e}{\omega_0^2-\omega^2+i\omega\gamma} E_0 \cos(\omega t)$ $\qquad \vec{F} = q\vec{v}\times\vec{B}$

$v = \dot{x} = \frac{-q_e/m_e}{\omega_0^2-\omega^2+i\omega\gamma} E_0 \sin(\omega t)$ $\qquad$ $\vec{v}$ is in the same direction as $\vec{E}$ so $\hat{v}\times\hat{B} = \hat{k}$ $\quad |\vec{v}\times\vec{B}| = vB$

$\vec{E} = \vec{E}_0 e^{i(\omega t)}$ at $z=0$ (this is inferred from where Eq. (3.65) comes from)

$\vec{B} = \frac{1}{c} E_0 e^{i(\omega t+\phi)} \hat{k}\times\hat{E}$

$Re\left[\vec{B}\right] = \frac{1}{c} E_0 \cos(\omega t+\phi)\,\hat{k}\times\hat{E}$

So: $\vec{F} = q_e \frac{(-q_e/m_e)E_0}{\omega_0^2-\omega^2+i\omega\gamma} \sin(\omega t)\cdot\frac{1}{c} E_0\cos(\omega t+\phi)\hat{k}$

$$\vec{F} = -\frac{q_e^2}{m_e}\frac{2}{c^2\epsilon_0}\frac{I}{\omega_0^2-\omega^2+i\omega\gamma}\sin(\omega t)\cos(\omega t+\phi)\hat{k}$$

If $\omega \ll \omega_0$ the electron will oscillate in phase with $\vec{E}$:

$$\vec{F}_{\omega\ll\omega_0} = -\frac{q_e^2}{m_e}\frac{1}{c^2\epsilon_0}\frac{I}{\omega_0^2-\omega^2+i\omega\gamma}\sin(2\omega t)\hat{k}$$

$\langle \vec{F}_{\omega\ll\omega_0}\rangle_\tau = 0$ Over many periods

If $\omega \approx \omega_0$ the electron will oscillate $\pi/2$ out-of-phase:

$$\vec{F} \approx +\frac{q_e^2}{m_e}\frac{2}{c^2\epsilon_0}\frac{I}{i\omega\gamma}\sin(2\omega t) \Rightarrow \langle\vec{F}\rangle_\tau = \frac{q_e^2}{m_e}\frac{1}{c^2\epsilon_0}\frac{I}{i\omega\gamma}\sin(2\omega t)$$

If $\omega \gg \omega_0$ the electron will oscillate π out-of-phase:

$\langle \vec{F}_{\omega\gg\omega_0}\rangle = 0$

$$\chi = \frac{q_e^2}{m_e}\frac{1}{\epsilon_0}\frac{1}{(\omega_0^2-\omega^2)+i\omega\gamma}$$

for $\omega \approx \omega_0$ $\quad \chi \approx \frac{q_e^2}{m_e}\frac{1}{i\epsilon_0\gamma\omega} = -i\frac{q_e^2}{m_e\epsilon_0}\frac{1}{\gamma\omega}$

$\tilde{n} = n_R - i n_I$ and $\alpha = 2\omega^{n_I}/c$ $\quad n \approx 1 + \chi/2$ $\quad n_I = \frac{1}{2}\frac{q_e^2}{m_e\epsilon_0}\frac{1}{\gamma\omega}$

So $\alpha = \frac{q_e^2}{m_e\epsilon_0 c}\frac{1}{\gamma}$

Recall assignment #1: $P = \frac{1}{c}I$ where P is the light pressure.

If we have an absorption coefficient α $\quad P = \frac{1}{c}I\alpha_L$ where α_L is the absorption for a length L. The total absorption will be α times the number of atoms in the path length, N.

$$P = \frac{1}{c}NI\alpha = \frac{q_e^2 N_L}{m_e\epsilon_0 c^2\gamma} I$$

Also $\langle|\vec{F}_{\omega=\omega_0}|\rangle_\tau = \frac{q_e^2}{m_e\epsilon_0 c^2\gamma} I$ for a single electron.

$\langle F\rangle_{tot} = N\langle F\rangle_e = \mathcal{N}V\langle F_e\rangle$ where $\mathcal{N}$ is the density

$P = \frac{1}{A}\langle F_{tot}\rangle = \mathcal{N}L\langle F_e\rangle$ and $\mathcal{N}L = N_L$ as defined above

So $P = \frac{q_e^2 N_L}{m_e\epsilon_0 c^2\gamma} I$