Phys 471 - Assignment #1 - Solutions

1. The wave equation is $\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t}$

$$\frac{\partial^2 \vec{E}}{\partial z^2} = -\vec{E}_o k^2 \sin(kz)\cos(\omega t) = -k^2 \vec{E}$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = -\vec{E}_o \omega^2 \sin(kz)\cos(\omega t) = -\omega^2 \vec{E}$$

so $$\frac{\partial^2 E}{\partial z^2} = \frac{k^2}{\omega^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{(\omega/k)^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

$\vec{E} = \vec{E}_o \sin(kz)\cos(\omega t)$ satisfies the wave equation with $v = \omega/k$

$$\sin\theta\cos\phi = \tfrac{1}{2}\left(\sin(\theta+\phi) + \sin(\theta-\phi)\right)$$

so $$\vec{E} = \vec{E}_o \tfrac{1}{2}\left[\sin(kz+\omega t) + \sin(kz-\omega t)\right]$$

$$= \tfrac{1}{2}\vec{E}_o\left[\sin(k(z+vt)) + \sin(k(z-vt))\right]$$

$$= \tfrac{1}{2}\vec{E}_o \sin(k(z+vt)) + \tfrac{1}{2}\vec{E}_o \sin(k(z-vt))$$

$\tfrac{1}{2}\vec{E}_o \sin(k(z+vt))$, $\tfrac{1}{2}\vec{E}_o \sin(k(z-vt))$ are functions describing left and right propagating wave respectively.

2. The wave propagates in the direction $(1,1,1)$ so $\hat{k} = \frac{1}{\sqrt{3}}(1,1,1)$

$\vec{E}$ is perpendicular to $\vec{k}$ so we have the constraint $\vec{E}_0 \cdot \vec{k} = 0$.
Because $\vec{E}_0$ is polarized in the x-y plane $E_{z_0} = 0$

So $\vec{E}_0 = (E_{x_0}, E_{y_0}, 0)$ $\quad \vec{E}_0 \cdot \vec{k} = (E_{x_0}, E_{y_0}, 0)\begin{pmatrix}1\\1\\1\end{pmatrix} = E_{x_0} + E_{y_0} = 0$

$E_{x_0} = -E_{y_0}$

$\vec{E}_0 = (E_{x_0}, -E_{x_0}, 0)$

We can also find $|\vec{E}_0|$:

$I = \frac{1}{2}\epsilon_0 c|\vec{E}_0|^2 = \frac{1}{2}\epsilon_0 c\left(E_{0x}^2 + (-E_{0x})^2\right) = \epsilon_0 c E_{0x}^2 = 1300 \, W/m^2$

$\Rightarrow E_{0x} = 700 \, N/C$

The expression for a plane wave is $\vec{E} = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$

$|\vec{k}| = 2\pi/\lambda = 9.9 \times 10^6 \, m^{-1}$

$\vec{E} = \vec{E}_0 \cos\left(|\vec{k}|(\vec{r}\cdot\hat{k} - ct)\right)$ with $\vec{r}$ being the spatial position.

where $\vec{E}_0 = 700 \, N/C \, (1,-1,0)$ $\quad |\vec{E}_0| = 990 \, N/C$
$|\vec{k}| = 9.9 \times 10^6 \, m^{-1}$
$\hat{k} = \frac{1}{\sqrt{3}}(1,1,1)$

3. The pressure exerted on a surface from light incident on it is given by the time average of the Poynting vector (which is the intensity of the incident light).

$$|\langle P \rangle_T| = \frac{1}{c}|\langle S \rangle_T| = I/c$$

The direction of the force is the same as the direction of the Poynting vector.

$$|\vec{F}| = PA = \frac{1}{c} IA = (3.0\times10^8 m/s)^{-1}(1300 W/m^2)(0.01 m^2) = \boxed{4.3\times10^{-8} N}$$

This assumes all light is absorbed.

If the plate is tilted the area "seen" by the incident light is reduced:

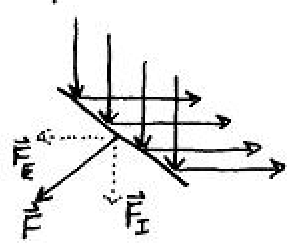
$$|\vec{F}| = \vec{P}\cdot\vec{A} = PA\cos(45°) = 4.3\times10^{-8}\cdot\cos(45°) = \boxed{3.1\times10^{-8} N}$$

The direction of the force is unchanged.

If the plate is 100% reflective, we can consider it to re-emit light in a direction perpendicular to the original direction:

$$|\vec{F}| = \sqrt{|\vec{F}_I|^2 + |\vec{F}_E|^2} = \sqrt{2|\vec{F}_I|^2} = \sqrt{2}(3.1\times10^{-8} N) = \boxed{4.3\times10^{-8} N}$$

The direction of the force will be along the normal of the plate.



4. Consider $\vec{\nabla}\times\vec{E}=\vec{B}$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\nabla(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E}$$ with no source terms $\vec{\nabla}\cdot\vec{E}=0$

$$\vec{\nabla}\times\vec{B}=-\nabla^2\vec{E}$$

$$\nabla^2\vec{E}=-\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) \qquad \vec{\nabla}\times\vec{B}=\mu_0\epsilon_0\frac{\partial E}{\partial t}$$

so $\nabla^2\vec{E}=-\mu_0\epsilon\ddot{\vec{E}}$

Because of the negative sign the wave equation is no longer satisfied.

If additionally $\vec{\nabla}\times\vec{H}=-\dot{\vec{D}}$, we get a second negative that cancels the first and the wave equation is satisfied.

If these two Maxwell equations read:

$$\vec{\nabla}\times\vec{E}=\dot{\vec{B}}$$
$$\vec{\nabla}\times\vec{H}=-\dot{\vec{D}}$$

the handedness of the electromagnetic wave would reverse.

5. Snell's law states: $\frac{\sin\theta_i}{\sin\theta_t} = \frac{n_2}{n_1}$

$n_2 = 1.70 \quad n_1 = 1$ so $\sin\theta_i = 1.7\sin\theta_t$

$\theta_t = \frac{1}{2}\theta_i \quad \Rightarrow \quad \sin\theta_i = 1.7\sin(\frac{1}{2}\theta_i)$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

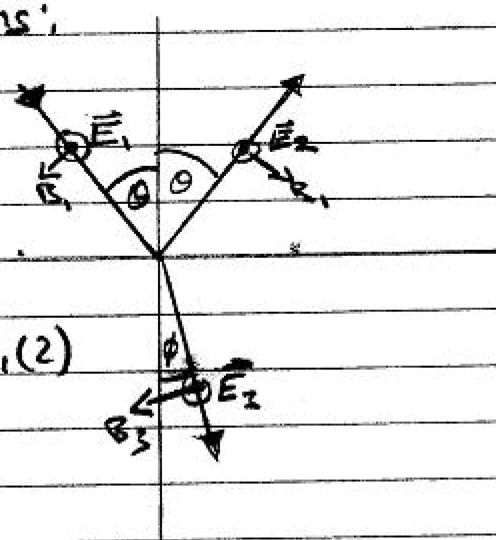
so $\sin\theta_i = \frac{1.7}{2}\frac{\sin\theta_i}{\cos(\frac{1}{2}\theta_i)}$

$\frac{1}{2}\theta_i = \arccos(1.7/2) = 0.55$

$\boxed{\theta_i = 1.1\,\mathrm{rad}}$ (64°)

3.2) Start with the $\perp$ (TE) case:

We have the boundary conditions:



$E_{\parallel_{in}} = E_{\parallel_{out}}$ and $H_{\parallel_{in}} = H_{\parallel_{out}}$

or
$$\begin{cases} \vec{E}_1 + \vec{E}_2 = \vec{E}_3 & \ldots (1) \\ -\vec{H}_1\cos\theta + \vec{H}_2\cos\theta = -\vec{H}_3\cos\phi & \ldots (2) \end{cases}$$
and

where $\vec{H} = \frac{\vec{B}}{\mu}$ ... (3)

define $r = \frac{E_2}{E_1}$ and $t = \frac{E_3}{E_1}$ ... (4)

(4) & (1) give $\vec{E}_1 + r\vec{E}_1 = t\vec{E}_1 \rightarrow 1 + r = t$ ... (5)

also $\cancel{q}vB = \cancel{q}E$ here $\rightarrow B_i = \frac{E_i}{v_i} = \frac{E_i}{\frac{1}{\sqrt{\mu_i\epsilon_i}}} = \sqrt{\mu_i\epsilon_i}\,E_i$ ↳ (6)

plugging (6) & (4) into (1): $-\frac{\sqrt{\mu_1\epsilon_1}}{\mu_1}\cancel{E_1}\cos\theta + \frac{\sqrt{\mu_1\epsilon_1}}{\mu_1} r\cancel{E_1}\cos\theta = -\frac{\sqrt{\mu_2\epsilon_2}}{\mu_2} t\cancel{E_1}\cos\phi$

or $-\sqrt{\frac{\epsilon_1}{\mu_1}}\cos\theta + r\sqrt{\frac{\epsilon_1}{\mu_1}}\cos\theta = -\sqrt{\frac{\epsilon_2}{\mu_2}}\cos\phi - r\sqrt{\frac{\epsilon_2}{\mu_2}}\cos\phi$

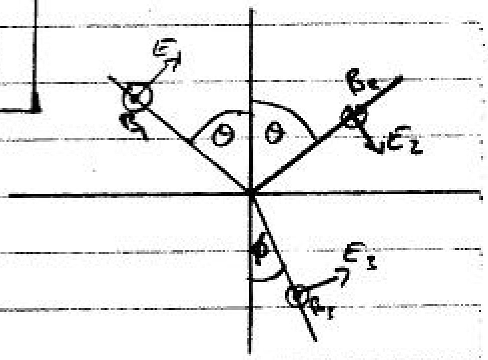
so $r\left(\sqrt{\frac{\epsilon_1}{\mu_1}}\cos\theta + \sqrt{\frac{\epsilon_2}{\mu_2}}\cos\phi\right) = \sqrt{\frac{\epsilon_1}{\mu_1}}\cos\theta - \sqrt{\frac{\epsilon_2}{\mu_2}}\cos\phi$

or
$$\boxed{r_{TE} = \frac{\sqrt{\frac{\epsilon_1}{\mu_1}}\cos\theta - \sqrt{\frac{\epsilon_2}{\mu_2}}\cos\phi}{\sqrt{\frac{\epsilon_1}{\mu_1}}\cos\theta + \sqrt{\frac{\epsilon_2}{\mu_2}}\cos\phi}}$$

similarily: $-\sqrt{\frac{\epsilon_1}{\mu_1}}\cos\theta + t\sqrt{\frac{\epsilon_1}{\mu_1}}\cos\theta - \sqrt{\frac{\epsilon_1}{\mu_1}}\cos\theta = -t\sqrt{\frac{\epsilon_2}{\mu_2}}\cos\phi$

or $t\left(\sqrt{\frac{\epsilon_1}{\mu_1}}\cos\theta + \sqrt{\frac{\epsilon_2}{\mu_2}}\cos\phi\right) = 2\sqrt{\frac{\epsilon_1}{\mu_1}}\cos\theta$

$$\text{or}\quad \boxed{t_{TE} = \frac{2\sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\theta}{\sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\theta + \sqrt{\frac{\varepsilon_2}{\mu_2}}\cos\phi}}$$



Now for the TM (||) case

here, from the diagram, we have:

$$\begin{cases} \vec{H}_1 - \vec{H}_2 = \vec{H}_3 & \dots\ (7) \\ E_1\cos\theta + E_2\cos\theta = E_3\cos\phi & \dots\ (8) \end{cases}$$

(7) says $\sqrt{\frac{\varepsilon_1}{\mu_1}}E_1 - \sqrt{\frac{\varepsilon_1}{\mu_1}}\,r E_1 = \sqrt{\frac{\varepsilon_2}{\mu_2}}\,t E_1$

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}(1-r) = \sqrt{\frac{\varepsilon_2}{\mu_2}}\,t \qquad \dots\ (9) \quad \Rightarrow 2 - t\frac{\cos\phi}{\cos\theta} = 1 + r$$

(8) gives $E_1\cos\theta + rE_1\cos\theta = tE_1\cos\phi \to t = (1+r)\frac{\cos\theta}{\cos\phi}$

so (9) reads $\sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\phi - r\sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\phi = \sqrt{\frac{\varepsilon_2}{\mu_2}}\cos\theta + r\sqrt{\frac{\varepsilon_2}{\mu_2}}\cos\theta$

or $r\left(\sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\phi + \sqrt{\frac{\varepsilon_2}{\mu_2}}\cos\theta\right) = \sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\phi - \sqrt{\frac{\varepsilon_2}{\mu_2}}\cos\theta$

$$\to \boxed{r_{TM} = \frac{\sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\phi - \sqrt{\frac{\varepsilon_2}{\mu_2}}\cos\theta}{\sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\phi + \sqrt{\frac{\varepsilon_2}{\mu_2}}\cos\theta}}$$

$\sqrt{\frac{\varepsilon_1}{\mu_1}}$

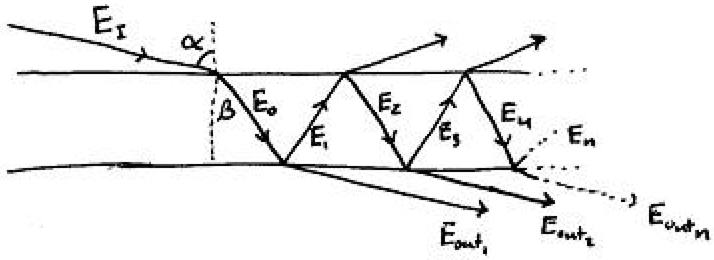
and $\to \sqrt{\frac{\varepsilon_1}{\mu_1}}\left(2 - t\frac{\cos\phi}{\cos\theta}\right) = \sqrt{\frac{\varepsilon_2}{\mu_2}}\,t \to 2\sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\theta - t\cos\phi = \sqrt{\frac{\varepsilon_2}{\mu_2}}\,t\cos\theta$

or $t\left(\sqrt{\frac{\varepsilon_2}{\mu_2}}\cos\theta + \sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\phi\right) = 2\sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\theta$

$$\text{so}\quad \boxed{t_{TM} = \frac{2\sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\theta}{\sqrt{\frac{\varepsilon_2}{\mu_2}}\cos\theta + \sqrt{\frac{\varepsilon_1}{\mu_1}}\cos\phi}}$$

Note that if $\mu = 1$, $\sqrt{\frac{\varepsilon_i}{\mu_i}} \to \sqrt{\varepsilon_i} = \frac{1}{v_i} = \frac{n_i}{c}$ and the original Fresnel equations are recovered.

7.



Assuming a transverse electic field:

$$t_1 = \frac{2n_1 \cos\alpha}{n_1\cos\alpha + n_2\cos\beta} \qquad \text{where} \quad |\vec{E}_0| = t_1 |\vec{E}_I|$$

t_1 is the transmission coefficient for light entering the plate

$$t_2 = \frac{2n_2\cos\beta}{n_2\cos\beta + n_1\cos\alpha} \qquad \cancel{|\vec{E}_1| = (1-t_2) = r_2} \quad |\vec{E}_1| = (1-t_2)|\vec{E}_0| = r_2 |\vec{E}_0|$$

$$E_{out} = E_{out_1} + E_{out_2} + \cdots = \sum_{n=1}^{\infty} E_{out_n} \qquad E_{out_n} = r_2^{2(n-1)} |\vec{E}_0| t_2$$

$$= \sum_{n=1}^{\infty} E_0 \, t_2 \, r_2^{2(n-1)}$$

$$= E_0 \, t_2 \sum_{n=0}^{\infty} r_2^{2n} \qquad \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

$$E_{out} = \frac{E_0 t_2}{1-r_2^2} = \frac{t_1 t_2 E_I}{1-r_2^2} \qquad 1 - r_2^2 = 2t_2 - t_2^2$$

$$E_{out} = \frac{t_1 E_I}{2 - t_2}$$

$$I_{out} = \frac{c\epsilon_0}{2} E_{out}^2 \qquad P_{out} = \frac{1}{2}\epsilon_0 E_{out}^2 \qquad P = \frac{1}{2}\epsilon_0 E_I^2$$

$$P_{out} = \frac{1}{2}\epsilon_0 \frac{t_1^2}{(2-t_2)^2} E_I^2 = \frac{t_1^2}{(2-t_2)^2} P$$

7 cont'

$$P_{out} = \frac{t_1^2}{(2-t_2)^2} P$$

$$P_{out} = \frac{4\cos^2\alpha}{(\cos\alpha + 1.5\cos\beta)^2}\left(2 - \frac{3\cos\beta}{1.5\cos\beta + \cos\alpha}\right)^{-2} P$$

$$\frac{\sin\alpha}{\sin\beta} = 1.5 \quad \Rightarrow \beta = \sin^{-1}\left(\tfrac{2}{3}\sin\alpha\right)$$

$$\cos\beta = \cos\left(\sin^{-1}\left(\tfrac{2}{3}\sin\alpha\right)\right) = \sqrt{1-\left(\tfrac{2}{3}\sin\alpha\right)^2}$$

$$P_{out} = \frac{4\cos^2\alpha}{\left(\cos\alpha + \frac{3}{2}\sqrt{1-\left(\frac{2}{3}\sin\alpha\right)^2}\right)^2}\left(2 - \frac{3\sqrt{1-\left(\frac{2}{3}\sin\alpha\right)^2}}{\frac{3}{2}\sqrt{1-\left(\frac{2}{3}\sin\alpha\right)^2} + \cos\alpha}\right)^{-2} P$$

$$P_{out} = \frac{4\cos^2\alpha}{\left(\left(\cos\alpha + \frac{3}{2}\sqrt{1-\left(\frac{2}{3}\sin\alpha\right)^2}\right)\left(2 - \frac{\sqrt{1-\left(\frac{2}{3}\sin\alpha\right)^2}}{\frac{1}{2}\sqrt{1-\left(\frac{2}{3}\sin\alpha\right)^2} + \frac{1}{3}\cos\alpha}\right)\right)^2} P$$