1. $FSR = \frac{c}{2l} = \frac{3.0\times10^8 m/s}{0.30m} = 1.0\times10^9 Hz$

$\mathcal{F} = \frac{\pi r}{1-r^2} = \frac{\pi\sqrt{0.99}}{1-0.99} = 313$

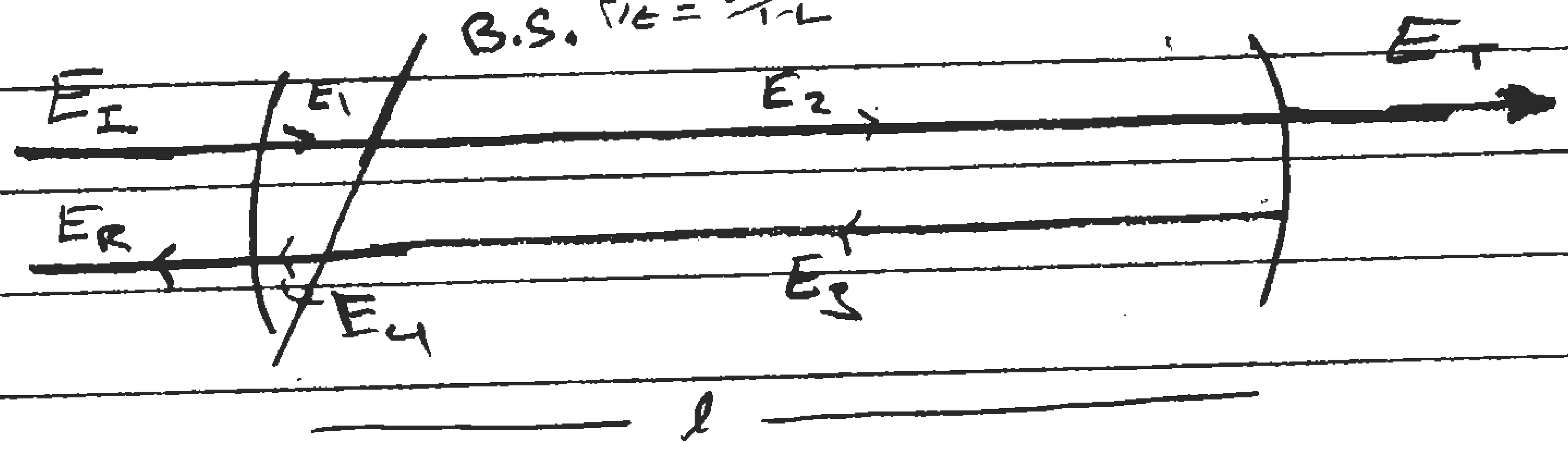
$FWHM = \frac{FSR}{\mathcal{F}} = \frac{1\times10^9 Hz}{313} = 3.2\times10^6 Hz$

To determine if $\delta\lambda$ is distinguishable we convert the FWHM to a wavelength (see 7.a) on assignment #4).

$\Delta\lambda_{min} = \frac{\lambda^2}{c} FWHM = \frac{(790nm)^2}{3.0\times10^8 m/s} \cdot 3.2\times10^6 Hz = 6.7\times10^{-15} m$

This is much smaller than $\delta\lambda$ (10^{-13}m) so $\delta\lambda$ is easily resolvable.

2.) A Fabry-Perot cavity containing an absorber can be modelled as follows



A beam splitter w/ transmission $1-L \equiv \eta$ and reflection L is placed between two mirrors of high reflectivity r, separated by a distance l.

In order to find T_{max}, T_{min}, and the FWHM of the cavity, we must write E_T in terms of E_I, r, and t.

Now with respect to the left most mirror,

$t = \sqrt{1-r^2}$

$$E_T = tE_2 e^{ikl}$$ ... where the phase has been picked up over the length of the cavity

now $E_2 = \eta E_1$ and,

$$E_1 = tE_I + rE_4 \rightarrow E_2 = \eta tE_I + \eta r E_4$$

we also see that $E_4 = \eta E_3$, and

$$e^{-ikl} E_3 = rE_2 e^{ikl}$$ ... where the direction of k and the phase shift on E_2 has been taken into account.

$$\rightarrow E_3 = rE_2 e^{2ikl}$$

$$\rightarrow E_4 = \eta r E_2 e^{2ikl}$$

Combining what we have so far:

$$E_2 = \eta t E_I + \eta r(\eta r E_2 e^{2ik\ell})$$

so $E_2(1-\eta^2 r^2 e^{2ik\ell}) = \eta t E_I$, or

$$E_2 = \frac{\eta t E_I}{1-\eta^2 r^2 e^{i2k\ell}}$$, which, since $E_T = t e^{ik\ell} E_2$, finally gives:

$$E_T = \frac{\eta t^2 e^{ik\ell}}{1-\eta^2 r^2 e^{i2k\ell}} E_I$$ or $t = \frac{\eta t^2 e^{ik\ell}}{1-\eta^2 r^2 e^{i2k\ell}}$

Maximum transmission then occurs when the denominator is minimal, i.e. when $e^{i2k\ell} = 1 \rightarrow 2k\ell = 2m\pi \rightarrow \ell = \frac{m\lambda}{2}$

then $t = \frac{\eta t^2 e^{im\pi = (-1)^m}}{1-\eta^2 r^2} \rightarrow |t|^2_{max} = \left(\frac{\eta(1-r^2)}{1-\eta^2 r^2}\right)^2$

note that when $\eta = 1$ (no attenuation) then $|t_{max}|^2 = \frac{1-r^2}{1-r^2} = 1$

i.e. we have total transmission.

Minimum transmission then occurs when $e^{2ik\ell} = -1 = e^{i(2m+1)\pi}$

so $2ik\ell = (2m+1)i\pi$ or $\ell = \frac{2m+1}{4}\lambda$

in this case: $|t| = \frac{\eta t^2 |\pm 1|}{1+\eta^2 r^2} \rightarrow |t_{min}|^2 = \frac{\eta(1-r^2)}{1+\eta^2 r^2}$

note as well, that when $\eta \rightarrow 1$, $|t_{min}|^2 \rightarrow \left|\frac{1-r^2}{1+r^2}\right|^2$

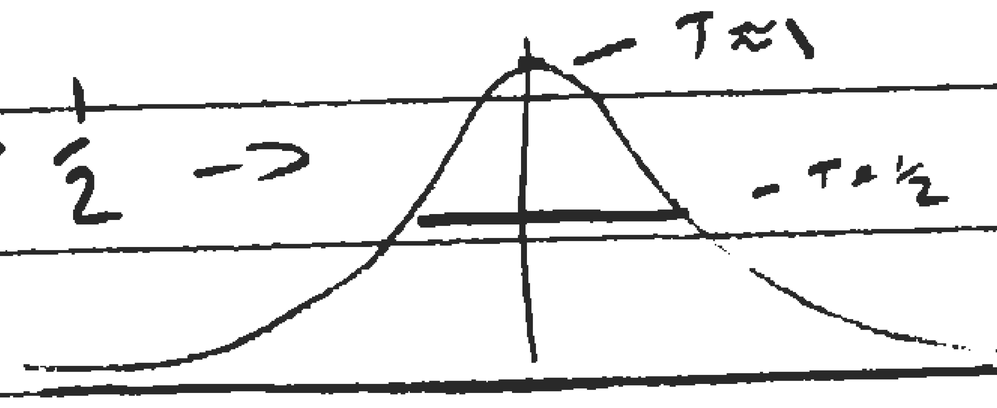
In order to find the FWHM of the cavity, consider the transmission intensity:

$$T = \frac{\eta^2(1-r^2)^2}{|1-\eta^2 r^2 e^{i2k\ell}|^2} = \frac{\eta^2(1-r^2)^2}{1+\eta^4 r^4 - 2\eta^2 r^2 \cos(2k\ell)}$$

$$= \frac{\eta^2(1-r^2)^2}{1+(\eta r)^4 - 2(\eta r)^2 + 4(\eta r)^2 \sin^2(k\ell)}$$

since $k\ell \ll 1$, $(\sin(k\ell))^2 \approx k^2\ell^2$

$$\approx \frac{\eta^2(1-r^2)^2}{1+(\eta r)^4 + 2(2(k\ell)^2 - 1)(\eta r)^2} = \frac{1}{2} \rightarrow$$



$$\rightarrow 2\eta^2 - 4\eta^2 r^2 + 2\eta^2 r^4 = 1 + (\eta r)^4 + 4\eta^2 r^2 k^2 \ell^2 - 2\eta^2 r^2$$

$$\rightarrow \frac{1}{2r^2} - \frac{1}{2} + \frac{r^2}{2} = \frac{1}{4\eta^2 r^2} + \frac{\eta^2 r^2}{4} + k_{\frac{1}{2}}^2 \ell^2 \cancel{- \frac{1}{2}}$$

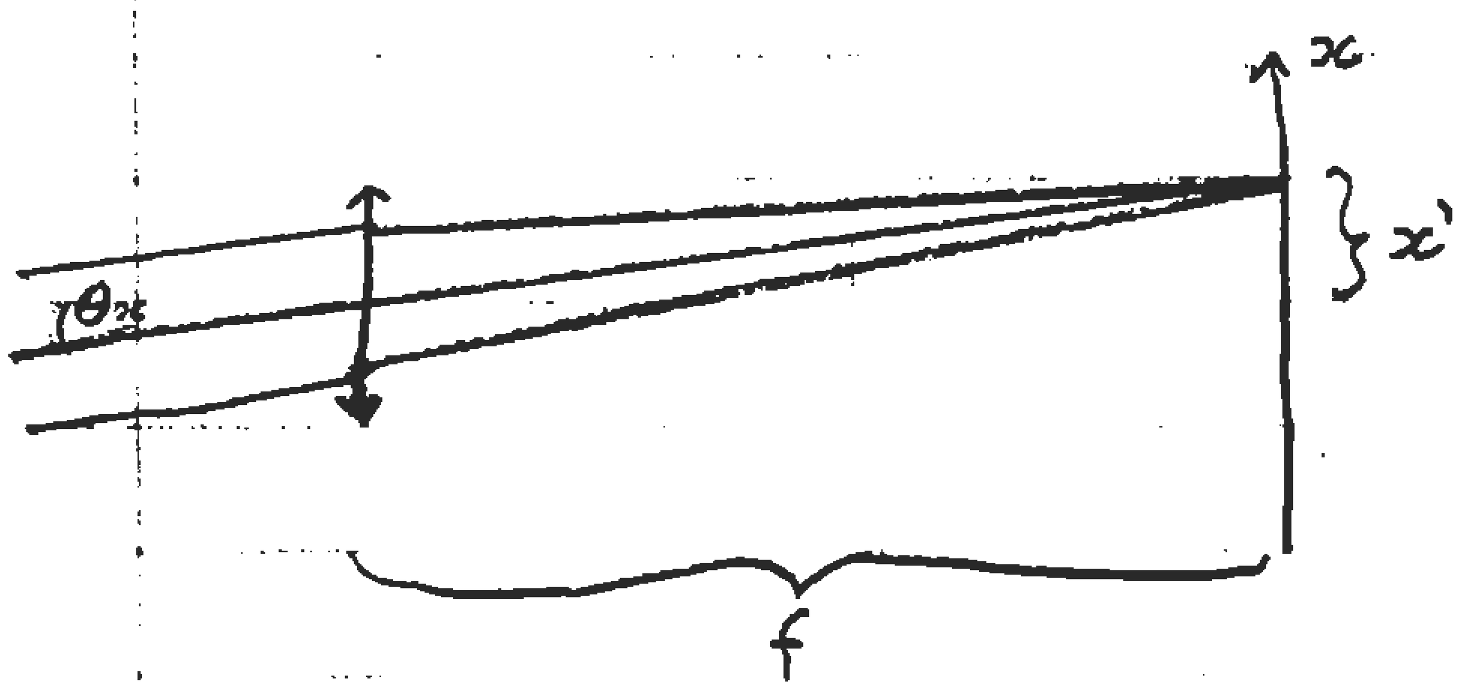
$$\frac{1}{\ell^2}\left(\frac{1-r^2+r^4}{2r^2} - \frac{1}{4r^2\eta^2} - \frac{\eta^2 r^2}{4}\right) = k_{\frac{1}{2}}^2$$

$$\rightarrow \Delta k_{FWHM} = \sqrt{\frac{2\eta^2 - 2\eta^2 r^2 + 2\eta^2 r^4 - 1 - \eta^2 r^2}{4\ell^2 r^2 \eta^2}}$$

∴ using $\Delta\omega_{FWHM} = c\Delta k_{FWHM}$

$$\rightarrow \Delta\omega_{FWHM} = \frac{(\eta^2(2-3r^2+2r^4)-1)^{\frac{1}{2}} c}{2\ell r\eta}$$

3. All parallel incoming rays will converge at a point in the focal plane given by the angle that the rays make with respect to the optical axis. This angle depends on the x and y components of the $\vec{k}$ vector: $k_x = \sin\theta_x|\vec{k}|$, $k_y = \sin\theta_y|\vec{k}|$ where θ_x, θ_y are angles with respect to the optical axis.



$f \tan\theta_x = x'$

Using the small angle approximation: $k_x \approx \theta_x |\vec{k}|$ and $x' \approx f\theta_x$

so $x' \approx \dfrac{f k_x}{|\vec{k}|} = \dfrac{f\lambda}{2\pi} k_x$ similarily $y' = \dfrac{f\lambda}{2\pi} k_y$

So, in the focal plane of the lens the electric field is given by the fourier components of the electric field at the lens.

Specifically $E(x', y') = \tilde{E}(\alpha k_x, \alpha k_y)$ where $\alpha = \dfrac{f\lambda}{2\pi}$

$$I(x', y') \propto |E(x', y')|^2 = |\tilde{E}(\alpha k_x, \alpha k_y)|^2$$

4. a) The pattern is a convolution of three delta functions with the slit.

$$E(x,y) = f_1 * f_2 \qquad f_1 = \begin{cases} 1 & \text{for } -b/2 \leq x \leq b/2 ,\ -c/2 \leq y \leq c/2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_2 = \delta(x+a) + \delta(x) + \delta(x-a)$$

$$\tilde{f}_1 = \int_{-b/2}^{b/2} \int_{-c/2}^{c/2} e^{-ik_x x} e^{-ik_y y}\, dy\, dx = bc \operatorname{sinc}\left(\tfrac{1}{2} b k_x\right) \operatorname{sinc}\left(\tfrac{1}{2} c k_y\right)$$

$$\tilde{f}_2 = \int_{-\infty}^{\infty} \left(\delta(x+a) + \delta(x) + \delta(x-a)\right) e^{-ik_x x}\, dx = e^{ik_x a} + 1 + e^{-ik_x a} = 1 + 2\cos(ak_x)$$

$$\tilde{E}(k_x, k_y) = E_0 \tilde{f}_1 \cdot \tilde{f}_2 = bc\,(1 + 2\cos(ak_x)) \operatorname{sinc}(\tfrac{1}{2} b k_x) \operatorname{sinc}(\tfrac{1}{2} c k_y)$$

$$I \propto |\tilde{E}(k_x, k_y)|^2 = E_0^2 b^2 c^2 (1 + 2\cos(ak_x))^2 \operatorname{sinc}^2(\tfrac{1}{2} b k_x) \operatorname{sinc}^2(\tfrac{1}{2} c k_y)$$

similarily to the last problem: $k_x \approx \dfrac{2\pi}{\lambda d} x' = 1.18 \times 10^6\, m^{-2} x'$

$$I = I_0 b^2 c^2 \left(1 + 2\cos\left(a \left(\tfrac{2\pi}{\lambda d}\right) x'\right)\right)^2 \operatorname{sinc}^2\left(\tfrac{1}{2} b \left(\tfrac{2\pi}{\lambda d}\right) x'\right) \operatorname{sinc}^2\left(\tfrac{1}{2} c \left(\tfrac{2\pi}{\lambda d}\right) y'\right)$$

$$I = I_0 (0.04\,mm^4)(1 + 2\cos(354\, m^{-1} x'))^2 \operatorname{sinc}^2(36\, m^{-1} x') \operatorname{sinc}^2(42\, m^{-1} y')$$

b) For a simple circular aperture:

$$\tilde{E}(k_x, k_y) = E_0 \int_{r=0}^{a} \int_{\phi=0}^{2\pi} e^{-i(k_x r \sin\phi + k_y r \cos\phi)} \, r\,dr\,d\phi$$

The details of this integration can be found in Hecht.

From this we get $E(\theta) = E_0 \left[\dfrac{2J_1(ka\sin\theta)}{ka\sin\theta} \right]$

where θ is an angle with respect to the optical axis.

For the pattern given in this problem we get:

$$\tilde{E}(k_x, k_y) = E_0 \left\{ \int_{r=0}^{a/2} \int_{\phi=0}^{2\pi} f(k_x, k_y, r, \phi)\, r\,dr\,d\phi + \int_{r=b/2}^{c/2} \int_{\phi=0}^{2\pi} f(k_x, k_y, r, \phi)\, r\,dr\,d\phi \right\}$$

where $f(k_x, k_y, r, \phi) = e^{-i(k_x r \sin\phi + k_y r\cos\phi)}$

the integral from $b/2$ to $c/2$ can be broken up:

$$\int_{r=b/2}^{c/2} \rightarrow \int_{r=0}^{c/2} - \int_{r=0}^{b/2}$$

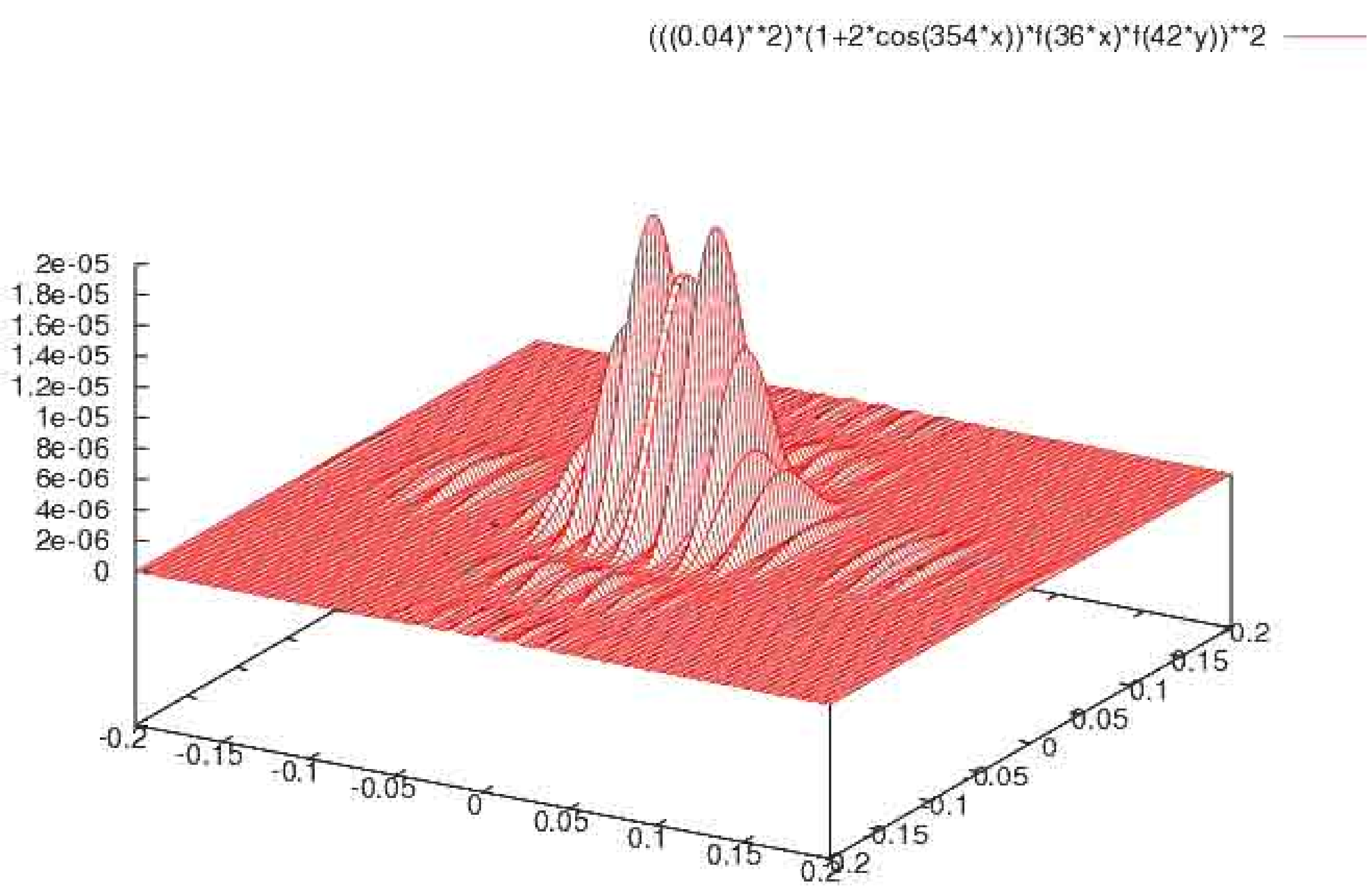
The solution can then be written:

$$E(\theta) = 2E_0 \left\{ \frac{J_1(\frac{1}{2}ak\sin\theta)}{\frac{1}{2}ak\sin\theta} + \frac{J_1(\frac{1}{2}ck\sin\theta)}{\frac{1}{2}ck\sin\theta} - \frac{J_1(\frac{1}{2}bk\sin\theta)}{\frac{1}{2}bk\sin\theta} \right\}$$

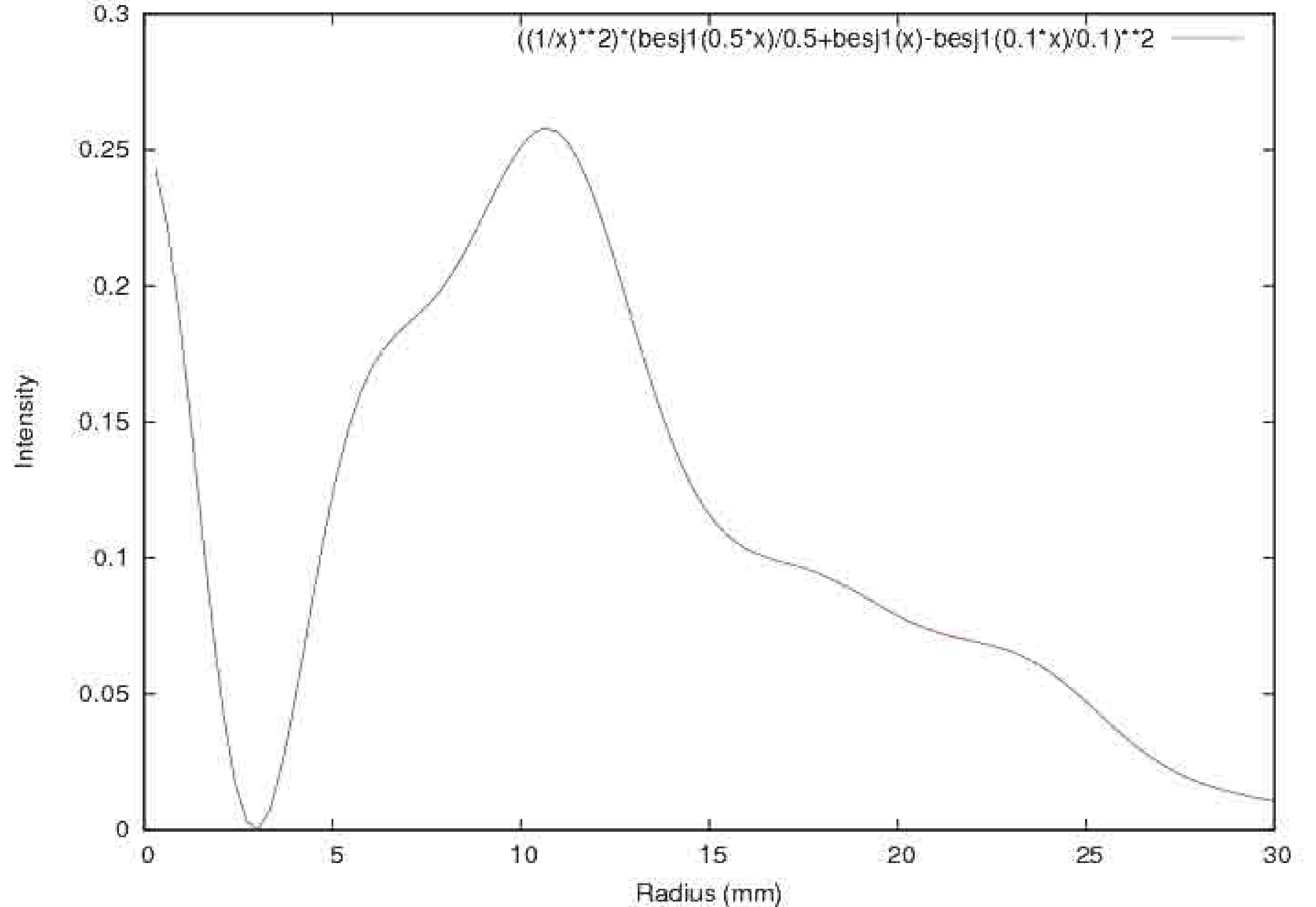
$r' = L\sin\theta \Rightarrow \sin\theta = r'/L$ where $L = 10\,\text{m}$

$$E(r') = 4E_0 \frac{L}{k} \frac{1}{r'} \left\{ \frac{J_1(\frac{1}{2}\frac{a}{L}kr')}{a} + \frac{J_1(\frac{1}{2}\frac{c}{L}kr')}{c} - \frac{J_1(\frac{1}{2}\frac{b}{L}kr')}{b} \right\}$$

So $I(r') = I_0 \left(\dfrac{L\lambda}{2}\right)^2 \left(\dfrac{1}{r'}\right)^2 \left\{ \dfrac{J_1(\frac{1}{2}\frac{a}{L}kr')}{a} + \dfrac{J_1(\frac{1}{2}\frac{c}{L}kr')}{c} - \dfrac{J_1(\frac{1}{2}\frac{b}{L}kr')}{b} \right\}^2$



(((0.04)**2)*(1+2*cos(354*x))*f(36*x)*f(42*y))**2
2e-05
1.8e-05
1.6e-05
1.4e-05
1.2e-05
1e-05
8e-06
6e-06
4e-06
2e-06
0
-0.2
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
0.2
-0.2
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
0.2


((1/x)**2)*(besj1(0.5*x)/0.5+besj1(x)-besj1(0.1*x)/0.1)**2
Intensity
0.3
0.25
0.2
0.15
0.1
0.05
0
0
5
10
15
20
25
30
Radius (mm)

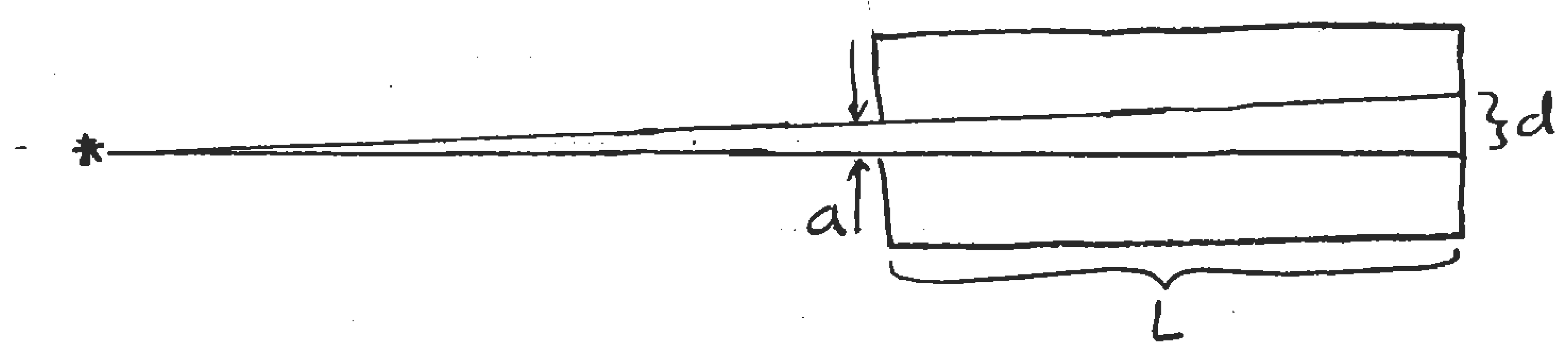
5. a) $\mathcal{F}_x[g(x+a)] = e^{iaw}\mathcal{F}(\omega)$

A shift in the negative x-direction will generate a phase shift (if the pattern is really at infinity).

b) $\mathcal{F}_x[g(ax)] = \frac{1}{|a|}\mathcal{F}\left(\frac{\omega}{a}\right)$

The pattern will be extended in the x direction and will shrink in the y direction.

6.



Consider an object at infinity:

The geometric spot size will be $d_g = a$

The diffraction limit is $\theta_{min} \approx \frac{1}{2} \lambda / a$

The spot size due to diffraction is $d_d \approx 2L \tan(\theta_{min}) \approx L\lambda/a$

For the optimal pinhole size $d_g = d_d$

$\Rightarrow L\lambda/a = a$

So $a = \sqrt{L\lambda}$

For 500nm light $a \approx 0.7mm$

7. The resolving power of a diffraction grating is proportional to the width of the grating as well as proportional to the slit size (due to the geometric spot size). We want to minimize the slit size without losing light so the slit should create a pattern of size D, 1m away.

$$\theta \approx \lambda/a \qquad \tan\theta = \frac{D}{L} \approx \theta$$

$$\Rightarrow \lambda/a \approx D/L \qquad a = \lambda L/D$$

So $a \approx 5.5\,\mu m$

$R \approx \dfrac{2\,Nd \sin\theta_i}{\lambda}$ $\quad Nd = 10\,cm$ (number of slits × spacing)
take $\theta_i = 45^\circ$ $\sin(45^\circ) = 1/\sqrt{2}$, $\lambda = 550\,nm$

$R \approx 2.6 \times 10^5$

$$\frac{\lambda}{\Delta\lambda} = R \Rightarrow \Delta\lambda = 0.14\,nm$$

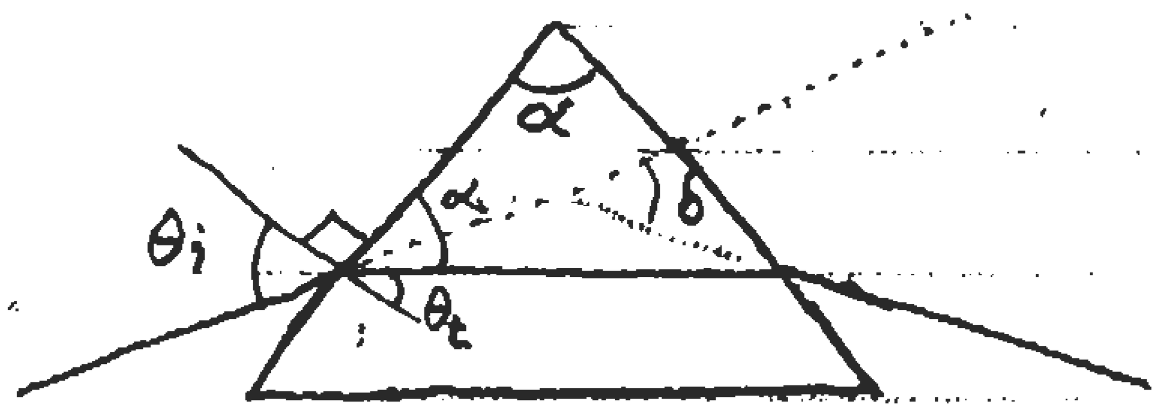
?

Alternately, we could argue that we want the angular uncertainty due to the slit width to be equal to the diffraction condition:

$$\theta \approx \lambda/a \quad \text{and} \quad \theta \approx D/L$$

So $a \approx \dfrac{\lambda L}{D}$ which is consistent.

8. Consider a prism in the minimum deviation configuration:



$\sin\theta_i = n\sin\theta_t$

and $\theta_t = 90° - \alpha$

for an equilateral prism $\alpha = 60°$

$$\delta = 2\cdot(\theta_i - \theta_t) = 2\cdot(\theta_i - \pi/6)$$

$$\sin\theta_i = n\sin(30°) = \tfrac{1}{2}n$$

$$\delta(\lambda) = 2\cdot(\sin^{-1}(\tfrac{1}{2}n(\lambda) - \pi/6)$$

$$\frac{\partial\delta(\lambda)}{\partial\lambda} = \frac{1}{\sqrt{1-\frac{1}{4}n^2}}\frac{\partial n}{\partial\lambda}$$

$\Delta\lambda = \Delta\delta\left(\frac{\partial\delta}{\partial\lambda}\right)^{-1}$ and $\Delta\delta \approx \frac{\lambda}{D}$ where D is the side length.

So $\frac{\Delta\lambda}{\lambda} \approx \frac{1}{D}\left(\frac{\partial\delta}{\partial\lambda}\right)^{-1}$ $\quad R = \frac{\lambda}{\Delta\lambda}$

so $R \approx D\left(\frac{\partial\delta}{\partial\lambda}\right) = \frac{D}{\sqrt{1-\frac{1}{4}n^2}}\left|\frac{\partial n}{\partial\lambda}\right|$ $\quad \left|\frac{\partial n}{\partial\lambda}\right| = \frac{|1.7328-1.7076|}{|486.1\text{nm}-656.3\text{nm}|} = 1.5\times10^{5}\text{m}$

$$R \approx \frac{0.1\text{m}}{\sqrt{1-\frac{1}{4}(n_{avg})^2}}1.5\times10^{5}\text{m}$$

$$R \approx 2\times10^4$$

9. The diffraction pattern will be generated by a pattern that can be considered to be a convolution of one period of the phase grating with a dirac comb.

The diffraction pattern will be $I_1(\theta) = \left(\mathcal{F}[E_0(x)]\right)^2 \left(\frac{\sin(N\alpha)}{\sin(\alpha)}\right)^2$

Where $\left(\frac{\sin(N\alpha)}{\sin(\alpha)}\right)^2$ is the fourier transform of N delta functions and $\alpha = \frac{1}{2} k d \sin(\theta)$. [from Hecht pg. 460]

For one period $\vec{E} = \vec{E}_0 \begin{cases} e^{i\phi/2} & \text{for } 0 \leq x \leq d/2 \\ e^{-i\phi/2} & \text{for } -d/2 \leq x < 0 \end{cases}$ $\phi = ka(n_1 - n_2)$

$$\mathcal{F}(\vec{E}) = \vec{E}_0 \left\{ \int_{-d/2}^{0} dx\, e^{-ik_x x} e^{-i\phi/2} + \int_{0}^{d/2} dx\, e^{-ik_x x} e^{i\phi/2} \right\}$$

$$= \vec{E}_0 \left\{ e^{-i\phi/2} \frac{i}{k_x}\left(1 - e^{i\frac{d}{2}k_x}\right) + e^{i\phi/2} \frac{i}{k_x}\left(e^{-i\frac{d}{2}k_x} - 1\right) \right\}$$

$$= \vec{E}_0 \frac{i}{k_x} \left\{ e^{-i\phi/2} - e^{i\phi/2} - e^{i(\frac{d}{2}k_x - \phi/2)} + e^{-i(\frac{d}{2}k_x - \phi/2)} \right\}$$

$$= \vec{E}_0 \frac{i}{k_x} \left\{ -2i\sin(\phi/2) - 2i\sin(\tfrac{d}{2}k_x - \phi/2) \right\}$$

$$= \vec{E}_0 \frac{1}{k_x} \left\{ 2\sin(\phi/2) + 2\sin(\tfrac{d}{2}k_x - \phi/2) \right\} \qquad k_x = k\sin\theta$$

$$I_2(\theta) = |\mathcal{F}(\vec{E})|^2 = I_0 \left(\frac{1}{k\sin\theta}\right)^2 4\left(\sin^2(\phi/2) + 2\sin(\phi/2)\sin(\tfrac{d}{2}k\sin\theta - \phi/2) + \sin^2(\tfrac{d}{2}k\sin\theta - \phi/2)\right)$$

$$= I_0 \left(\frac{d/2}{\alpha}\right)^2 4\left(\sin^2(\phi/2) + \cos(\alpha+\phi) - \cos\alpha + \sin^2(\alpha - \phi/2)\right) \qquad \alpha = \tfrac{d}{2}k\sin\theta$$

$$I(\theta) = I_1 \cdot I_2 = I_0 \left(\frac{\sin(N\alpha)}{\sin(\alpha)}\right)^2 \left(\frac{d}{\alpha}\right)^2 \left(\sin^2(\phi/2) + \cos(\alpha+\phi) - \cos(\alpha) + \sin^2(\alpha - \phi/2)\right)$$

from $\left(\frac{\sin(N\alpha)}{\sin(\alpha)}\right)^2$ we get principle maxima when $\alpha = \pm n\pi$ within the envelope of $\left(\sin^2(\phi/2) + \cos(\alpha+\phi) - \cos(\alpha) + \sin^2(\alpha - \phi/2)\right)$

The ratio $\frac{I_{first}}{I_{zeroth}}$ is given by $\frac{I(\alpha=\pi)}{I(\alpha=0)}$

$$\frac{I_{first}}{I_{zeroth}} = \frac{\sin^2(\phi/2) - \cos(\phi) + 1 + \sin^2(\phi/2)}{\sin^2(\phi/2) + \cos(\phi) - 1 + \sin^2(\phi/2)}$$

$$\frac{I_{first}}{I_{zeroth}} = \frac{2\sin^2(\phi/2) + 1 - \cos(\phi)}{2\sin^2(\phi/2) - 1 + \cos(\phi)}$$