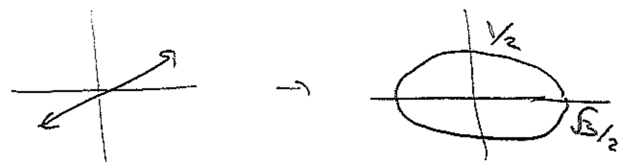
PHYS 443 Winter 2017

Midterm examination

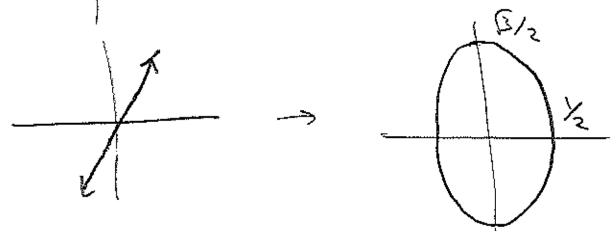
Solutions

1 $\alpha = 0$: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\alpha = 30°$ $\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \to \begin{pmatrix} \sqrt{3}/2 \\ i/2 \end{pmatrix}$



$\alpha = 60°$ $\begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} \to \begin{pmatrix} 1/2 \\ i\sqrt{3}/2 \end{pmatrix}$



$\alpha = 90°$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ i \end{pmatrix}$

2 a) I: $|H\rangle$; II: $\cos 2\alpha\,|H\rangle + \sin 2\alpha\,|V\rangle = |2\alpha\rangle$

b) I: $\{|2\alpha\rangle, |\tfrac{\pi}{2}+2\alpha\rangle\}$; II: $\{|H\rangle, |V\rangle\}$

c) I: $|\langle 2\alpha|H\rangle|^2 = \cos^2 2\alpha$; $|\langle\tfrac{\pi}{2}+2\alpha\,|H\rangle|^2 = \sin^2 2\alpha$

II: $|\langle H|2\alpha\rangle|^2 = \cos^2 2\alpha$; $|\langle V|2\alpha\rangle|^2 = \sin^2 2\alpha$

d) I: $|2\alpha\rangle\langle 2\alpha| - |\tfrac{\pi}{2}+2\alpha\rangle\langle\tfrac{\pi}{2}+2\alpha|$

$$= \begin{pmatrix} \cos^2 2\alpha & \cos 2\alpha \sin 2\alpha \\ \cos 2\alpha \sin 2\alpha & \sin^2 2\alpha \end{pmatrix} - \begin{pmatrix} \sin^2 2\alpha & -\cos 2\alpha \sin 2\alpha \\ -\cos 2\alpha \sin 2\alpha & \cos^2 2\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2 2\alpha - \sin^2 2\alpha & -2\cos 2\alpha \sin 2\alpha \\ 2\cos 2\alpha \sin 2\alpha & \sin^2 2\alpha - \cos^2 2\alpha \end{pmatrix} = \begin{pmatrix} \cos 4\alpha & \sin 4\alpha \\ \sin 4\alpha & -\cos 4\alpha \end{pmatrix}$$

II: $|H\rangle\langle H| - |V\rangle\langle V| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

e) I: $\langle H|2\alpha\rangle\langle 2\alpha|H\rangle - \langle H|\tfrac{\pi}{2}+2\alpha\rangle\langle\tfrac{\pi}{2}+2\alpha|H\rangle = \cos^2 2\alpha - \sin^2 2\alpha = \cos 4\alpha$

II: $\langle 2\alpha|H\rangle\langle H|2\alpha\rangle - \langle 2\alpha|V\rangle\langle V|2\alpha\rangle = \cos^2 2\alpha - \sin^2 2\alpha = \cos 4\alpha$

$\boxed{3}$ $[\sigma_x + i\sigma_y, \sigma_x - i\sigma_y] = -i[\sigma_x\, \sigma_y] + i[\sigma_y\, \sigma_x] = -i(2i\sigma_z) + i(-2i\sigma_z) = 4\sigma_z$

$[\sigma_x^2 + \sigma_y^2 + \sigma_z^2, \sigma_z] = [\sigma_x^2, \sigma_z] + [\sigma_y^2, \sigma_z] =$

$= \sigma_x[\sigma_x\, \sigma_z] + [\sigma_x\, \sigma_z]\sigma_x + \sigma_y[\sigma_y\, \sigma_z] + [\sigma_y\, \sigma_z]\sigma_y$

$= -2i\sigma_x\sigma_y - 2i\sigma_y\sigma_x + 2i\sigma_y\sigma_x + 2i\sigma_x\sigma_y = 0$

$\boxed{4}$ a) $H_0 = E\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

b) $H = H_0 + V = E\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 2E|+\rangle\langle+| + 0|-\rangle\langle-|$

$U = e^{-iHt/\hbar} = e^{-2iEt/\hbar}|+\rangle\langle+| + |-\rangle\langle-|$

$= \frac{1}{2}e^{-2iEt/\hbar}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} e^{-2iEt/\hbar}+1 & e^{-2iEt/\hbar}-1 \\ e^{-2iEt/\hbar}-1 & e^{-2iEt/\hbar}+1 \end{pmatrix}$

c) $U|\psi(0)\rangle = \frac{1}{2}\begin{pmatrix} e^{-2iEt/\hbar}+1 \\ e^{-2iEt/\hbar}-1 \end{pmatrix}$

$pr_2 = |\langle v_1|\psi(t)\rangle|^2 = \frac{1}{4}|e^{-2iEt/\hbar} - 1|^2 =$

$= \frac{1}{4}|e^{-iEt/\hbar}|^2 |e^{-iEt/\hbar} - e^{iEt/\hbar}|^2 = \sin^2\frac{Et}{\hbar}$

$\boxed{5}$ a) $N = \sqrt{1^2 + 1^2 + 3^2} = \sqrt{11} \implies |\psi\rangle = \frac{1}{\sqrt{11}}(|HH\rangle + i|HV\rangle + 3|VV\rangle)$

b) $\sigma_x \otimes \sigma_z |\psi\rangle = (|H\rangle\langle V| + |V\rangle\langle H|) \otimes (|H\rangle\langle H| - |V\rangle\langle V|)|\psi\rangle$

$= \frac{1}{\sqrt{11}}(|VH\rangle - i|VV\rangle - 3|HV\rangle)$

c) $\langle R|\Psi\rangle = \frac{1}{\sqrt{22}}(\langle H| - i\langle V|)(|HH\rangle + i|HV\rangle + 3|VV\rangle)$

$= \frac{1}{\sqrt{22}}(|H\rangle + i|V\rangle - 3i|V\rangle) = \frac{1}{\sqrt{22}}(|H\rangle - 2i|V\rangle)$

$pr_R = \frac{1^2 + 2^2}{22} = \frac{5}{22}$

$\langle L|\psi\rangle = \frac{1}{\sqrt{22}}(\langle H| + i\langle V|)(|HH\rangle + i|HV\rangle + 3|VV\rangle)$

$= \frac{1}{\sqrt{22}}(|H\rangle + i|V\rangle + 3i|V\rangle) = \frac{1}{\sqrt{22}}(|H\rangle + 4i|V\rangle)$

$pr_L = \frac{1^2 + 4^2}{22} = \frac{17}{22}$