University of Calgary
Winter semester 2017

PHYS 443: Quantum Mechanics I

Homework assignment 6

Due April 12, 2017

Problem 6.1. For the displacement operator $\hat{D} \equiv \hat{D}_X(X_0, 0) = \hat{D}_{XP}(X_0, 0)$:

a) Find $\hat{D}^\dagger \hat{a} \hat{D}$ and $\hat{D}^\dagger \hat{a}^\dagger \hat{D}$;

b) Find $[a, \hat{D}]$ and $[a^\dagger, \hat{D}]$;
Hint: apply $\hat{D}$ to both sides of the result of part (a).

c) Find the Fock decomposition of the *displaced single-photon state* $\hat{D}\left|1\right\rangle$.
Hint: $\left|n\right\rangle = (\hat{a}^\dagger)^n \left|0\right\rangle / \sqrt{n!}$.

Problem 6.2. Calculate the reflection and transmission for scattering on a delta-potential $V(x) = W_0 \delta(x)$, with the energy $E > 0$. Compare your results with those obtained from Eqs. (3.80) in the lecture notes for an infinitely thin and high rectangular potential barrier ($L \to 0$, $V_0 = W_0/L$).

Problem 6.3. Consider coherent superpositions of coherent states $\left|S_\pm\right\rangle = \mathcal{N}_\pm(\left|\alpha\right\rangle \pm \left|-\alpha\right\rangle)$, where $\mathcal{N}_\pm$ are normalization factors and the alplitude α is real and positive[1].

a) Find $\mathcal{N}_\pm$.

b) Find the matrices (wavefunctions) of these states

- in the Fock basis;
- in the position basis;
- in the momentum basis.

c) Show that, for small amplitude α, these states can be approximated, up to the first two terms in the Fock decomposition, by states

$$\left|S_+\right\rangle \approx \hat{S}(r_+)\left|0\right\rangle;$$
$$\left|S_-\right\rangle \approx \hat{S}(r_-)\left|1\right\rangle$$

and find $r_\pm(\alpha)$ for which the approximation is optimal.

Problem 6.4. Consider the following state two harmonic oscillators:

$$\left|\psi\right\rangle = \alpha\left|0,0\right\rangle - \beta\left|1,1\right\rangle,$$

where α and β are real; $\alpha^2 + \beta^2 = 1$.

a) At which values of α and β does this state exhibit two-mode position-squeezing, i.e. the variance of $\hat{X}_A - \hat{X}_B$ is lower than that of the double-vacuum state?

[1] This state is dubbed the "Schrödinger kitten" because it is a superposition of two "classical" and potentially macroscopic coherent states, and yet is highly nonclassical. It is a subject of intense research because, by constructing such states with increasingly high amplitudes α, we may be able to find the boundaries of quantum physics — see Sec. 2.4.3 in the notes.

b) Answer the same question for the observable $\hat{P}_A - \hat{P}_B$.

Problem 6.5. A harmonic oscillator, initially in the vacuum state, evolves under the Hamiltonian $\hat{H} = \hbar r \hat{X}^2$, with a real and positive r, for time t.

a) Express the final quadrature operators $\hat{X}(t)$ and $\hat{P}(t)$ through the initial ones $\hat{X}(0)$ and $\hat{P}(0)$.

b) Find the mean and variance of the general *quadrature observable* $\hat{X}_\theta(t) = \hat{X}(t)\cos\theta \hat{+} P(t)\sin\theta$ for arbitrary angle θ.

c) Which angle θ_0 corresponds to the lowest variance of that observable?
Hint:

$$\sin(2x) = 2\sin x \cos x;$$
$$\cos(2x) = 1 - 2\sin^2 x = 2\cos^2 x - 1.$$

d) Write the uncertainty principle for the quadratures $\hat{X}_{\theta_0}$ and $\hat{X}_{\theta_0+\pi/2}$ and verify explicitly that it holds.
Hint:

$$\sin(\arctan x) = 1/\sqrt{1+x^2};$$
$$\cos(\arctan x) = x/\sqrt{1+x^2}.$$

e) Find the Fock decomposition of the state $|\psi\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|0\rangle$ for $rt \ll 1$ up to the two-photon term.

f) Find the variance of $\hat{X}_\theta$ in that state up to the first term in rt and verify consistency with part (b).
Hint: It may be convenient to express $\hat{X}_\theta$ in terms of $\hat{a}$ and $\hat{a}^\dagger$.

Parts (a)-(d) should be answered in the Heisenberg picture, parts (e)-(f) in the Schrödinger picture. Parts (d)-(f) are for extra credit.