University of Calgary
Winter semester 2017

PHYS 443: Quantum Mechanics I

Homework assignment 3

Due February 28, 2017

Problem 3.1. Write the uncertainty principle for observables $A = |H\rangle\langle H| + 3\,|H\rangle\langle V| + 3\,|V\rangle\langle H| + 2\,|V\rangle\langle V|$ and $\hat{B} = |R\rangle\langle R| + 3\,|L\rangle\langle L|$ and state $|\psi\rangle = |+\rangle$. Verify explicitly that it holds.

Problem 3.2. Consider the observable

$$\hat{\sigma}_\theta = |\theta\rangle\langle\theta| - \left|\frac{\pi}{2} + \theta\right\rangle\left\langle\frac{\pi}{2} + \theta\right|$$

(so that $\hat{\sigma}_{\theta=0} = \hat{\sigma}_z$ and $\hat{\sigma}_{\theta=\pi/4} = \hat{\sigma}_x$).

a) Find the matrix of $\hat{\sigma}_\theta$ in the canonical basis. **Hint:** denote $c = \cos 2\theta$, $s = \sin 2\theta$.

b) For observable $\hat{\sigma}_z \otimes \hat{\sigma}_\theta$:

   (a) calculate the matrix in the canonical basis $\{|HH\rangle, |HV\rangle, |VH\rangle, |VV\rangle\}$;

   (b) determine the eigenstates and eigenvalues (**hint:** you need not solve any equations);

   (c) calculate the expectation value and uncertainty in Bell state $|\Psi^-\rangle$.

Problem 3.3. A bireflingent material of length t with its optical axis oriented at angle α to horizontal enacts the following transformation:

$$|\alpha\rangle \to e^{ikn_e t}\,|\alpha\rangle\,;$$
$$|\pi/2 + \alpha\rangle \to e^{ikn_o t}\,|\pi/2 + \alpha\rangle\,,$$

where k is the wavenumber and $n_{o,e}$ are the ordinary and extraordinary refractive index.

a) Find the length t_0 for which this transformation corresponds to the half-wave plate.

b) Interpreting this transformation as quantum evolution under a certain Hamiltonian, find the matrix of this Hamiltonian in its eigenbasis and in the canonical basis. What are the energy eigenvalues?

Problem 3.4. The tensor product Hilbert space of Alice's and Bob's photons evolves under a Hamiltonian

$$\hat{H} = \hbar\omega(\hat{\sigma}_x \otimes \hat{\sigma}_x + \hat{\sigma}_y \otimes \hat{\sigma}_y + \hat{\sigma}_z \otimes \hat{\sigma}_z).$$

a) Write the Hamiltonian in the Dirac notation in terms of outer products of the canonical basis states $\{|HH\rangle, |HV\rangle, |VH\rangle, |VV\rangle\}$.

b) Find the 4×4 matrix of the Hamiltonian in the canonical basis.

c) Find the matrix of the evolution operator $e^{-iHt/\hbar}$.

d) Find the evolution of the state $|\psi(0)\rangle = |HV\rangle$.

Problem 3.5. Alice and Bob share two photons in polarization state

$$|\Psi\rangle = \frac{1}{\sqrt{14}}(2\,|HV\rangle + 3\,|VH\rangle + |VV\rangle).$$

a) Alice and Bob both perform measurements on their respective photons. Find the probabilities of all possible results.

b) Only Alice performs a polarization measurement on her photon. Find the probability of each outcome and the remotely prepared state of Bob's photon after the measurement. Apply each of the two alternative techniques to solve the problem in each basis:

- using the partial inner product;
- decomposing the initial state according to Eq. (2.16) in the lecture notes.

c) Suppose Bob does not know Alice's result. Based on part (b), give a verbal description of the state of Bob's photon after Alice's measurement.

d) Verify that the probability values found in parts (a) and (b) are consistent with each other.

Solve this problem for all the measurements performed in (i) diagonal and (ii) circular bases.