University of Calgary
Winter semester 2017

PHYS 443: Quantum Mechanics I

Homework assignment 2

Due February 7, 2017

Problem 2.1. Consider an operator $\hat{A}$ that performs the following transformation.

$$\hat{A}\left|H\right\rangle = i\left|H\right\rangle + 3\left|V\right\rangle; \tag{1}$$

$$\hat{A}\left|R\right\rangle = \sqrt{2}\left|V\right\rangle. \tag{2}$$

a) Find $\hat{A}\left|V\right\rangle$. Write your answer in the Dirac notation in terms of states $\left|H\right\rangle$, $\left|V\right\rangle$.

b) Express $\hat{A}$ in the Dirac notation in terms of outer products of states $\left|H\right\rangle$, $\left|V\right\rangle$.

c) Express $\hat{A}$ in the matrix notation in the canonical basis.

d) Find the matrix of $\hat{A}$ in the diagonal basis using $A_{ij} = \left\langle v_i \middle| \hat{A} \middle| v_j \right\rangle$.

e) Express $\hat{A}$ in the Dirac notation in terms of outer products of states $\left|+\right\rangle$, $\left|-\right\rangle$. Then express the diagonal polarization vectors in terms of the canonical polarization vectors. Simplify the result and check it to be identical to that of part (c).

f) Is $\hat{A}$ Hermitian? If not, what is its adjoint?

Problem 2.2. The operator associated with the quarter-wave plate with the optic axis oriented at angle α in the basis $\{\left|\alpha\right\rangle, \left|90^\circ + \alpha\right\rangle\}$ has matrix

$$\hat{A}_{\mathrm{QWP}}(\alpha) \simeq \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix};$$

Convert this matrix to the canonical basis using the method of inserting identity operator.

Problem 2.3. Using operator algebra, show that a quarter-wave plate oriented at any angle, when applied to a circular polarization state, generates a linear polarization state.

Problem 2.4. Operator $\hat{A}$ has the following matrix in the canonical basis.

$$\hat{A} \simeq \begin{pmatrix} 9 & 2 \\ 2 & 6 \end{pmatrix}$$

a) Present this operator in the form $\hat{A} = v_1 \left|v_1\right\rangle\left\langle v_1\right| + v_2 \left|v_2\right\rangle\left\langle v_2\right|$, where $\{\left|v_1\right\rangle, \left|v_2\right\rangle\}$ is an orthonormal basis. Find v_1, v_2, as well as the matrices of $\left|v_1\right\rangle$ and $\left|v_2\right\rangle$ in the canonical basis.

b) Write the matrices of outer products $\left|v_{1,2}\right\rangle\left\langle v_{1,2}\right|$ in the canonical basis and verify explicitly that $\hat{A} = v_1 \left|v_1\right\rangle\left\langle v_1\right| + v_2 \left|v_2\right\rangle\left\langle v_2\right|$.

c) Observable $\hat{A}$ is measured in the diagonally polarized state $\left|+\right\rangle$. What are the probabilities of possible outcomes?

d) Calculate the expectation value of the measurement result.

- using the definition of the expectation value from the probability theory;
- using the expression for the quantum mean.

Verify that the results are the same.

e) Calculate the variance of observable $\hat{A}$ in state $|+\rangle$.

Problem 2.5. Consider an apparatus for measuring the photon polarization that has the following properties:

- whenever the state $|\psi\rangle \simeq \begin{pmatrix} a_H \\ a_V e^{i\phi} \end{pmatrix}$ (where $a_{H,V}$ real and non-negative and ϕ is real; $a_H^2 + a_V^2 = 1$) enters the apparatus, it displays "1";
- whenever the state $|\psi_\perp\rangle$ enters the apparatus, it displays "-1";
- for photons with polarizations other than the above, it randomly displays one of these numbers with some probabilities.

a) Find the matrix of $|\psi_\perp\rangle$. **Hint:** Look for this state in the form $|\psi_\perp\rangle \simeq \begin{pmatrix} b_H \\ b_V e^{i\chi} \end{pmatrix}$

b) Find the matrix of the observable corresponding to this apparatus both in its eigenbasis and in the $\{|H\rangle, |V\rangle\}$ basis.

c) For which a_H, a_V and ϕ does this observable become each of the three Pauli operators?