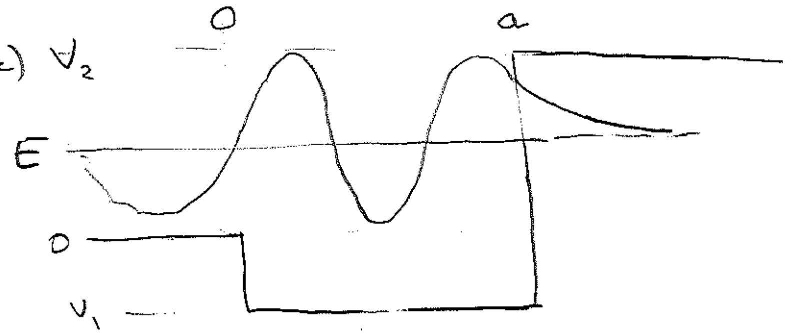
PHYS 443 Winter 2013

Final exam

Solutions

$\boxed{1}$ a)



Ae^{ikx} $\quad Be^{ik_0x}+Ce^{-ik_0x}$ $\quad De^{-\varkappa x}$

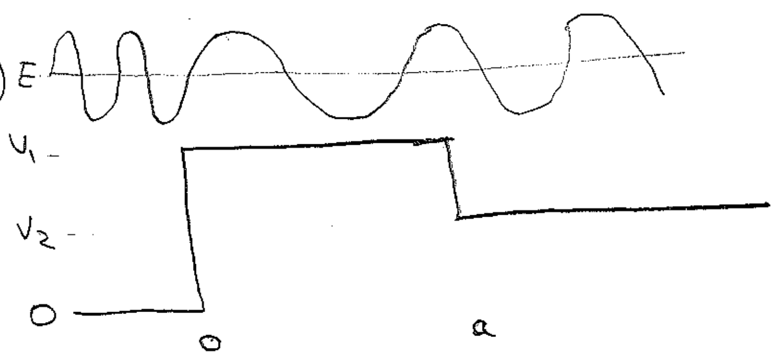
$$k_0 = \frac{\sqrt{2mE}}{\hbar}$$

$$k_1 = \frac{\sqrt{2m(E-V_1)}}{\hbar}$$

$$\varkappa = \frac{\sqrt{2m(V_2-E)}}{\hbar}$$

$$\begin{cases} A = B+C \\ k_1 A = k_0(B-C) \\ Be^{ik_0a}+Ce^{-ik_0a} = De^{-\varkappa a} \\ ik_0(Be^{ik_0a}+Ce^{-ik_0a}) = -\varkappa De^{-\varkappa a} \end{cases}$$

b)



$$k_2 = \frac{\sqrt{2m(E-V_2)}}{\hbar}$$

Ae^{ik_1x} $\quad Be^{ik_0x}+Ce^{-ik_0x}$ $\quad De^{ik_2x}$

$$A = B+C$$

$$k_1 A = k_0(B-C)$$

$$Be^{ik_0a}+Ce^{-ik_0a} = De^{ik_2a}$$

$$k_0(Be^{ik_0a}+Ce^{-ik_0a}) = k_2 De^{ik_2a}$$

$\boxed{2}$ $\hat{X} = \frac{a + a^+}{\sqrt{2}}$ $\hat{p} = \frac{a - a^+}{\sqrt{2}i}$

$$\langle m | \hat{a} | n \rangle = \sqrt{n}\, \delta_{m,n-1}$$

$$\langle m | a^+ | n \rangle = \sqrt{n+1}\, \delta_{m,n+1}$$

$$\hat{a} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & \sqrt{2} & & \\ & & 0 & \sqrt{3} & \\ & & & 0 & 2 \\ & & & & 0 \end{pmatrix} \qquad a^+ = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \sqrt{2} & 0 & & \\ & & \sqrt{3} & 0 & \\ & & & 2 & 0 & \ddots \end{pmatrix}$$

$$\hat{X} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & \sqrt{2} & & \\ & \sqrt{2} & 0 & \sqrt{3} & \\ & & \sqrt{3} & 0 & 2 \\ & & & 2 & 0 & \ddots \end{pmatrix} \qquad \hat{p} = \frac{1}{i\sqrt{2}} \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & \sqrt{2} & & \\ & -\sqrt{2} & 0 & \sqrt{3} & \\ & & -\sqrt{3} & 0 & 2 \\ & & & -2 & 0 & \ddots \end{pmatrix}$$

$\boxed{3}$ a) $\int |\psi(x)|^2 dx = N^2 \int_0^a x^2 dx = N^2 a^3/3$

$N = \sqrt{3}\, a^{-3/2}$

b) $\langle x \rangle = \int x |\psi(x)|^2 dx = N^2 \int_0^a x^3 dx = N^2 a^4/4 = \frac{3}{4} a$

$\langle x^2 \rangle = N^2 a^5/5 = \frac{3}{5} a^2$

$\langle \Delta x^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 = \frac{3}{5} a^2 - \frac{9}{16} a^2 = \frac{3}{80} a^2$

c) $\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \psi(x) e^{-ipx/\hbar} dx$

$= \frac{1}{\sqrt{2\pi\hbar}} N \int_0^a x e^{-ipx/\hbar} dx$

$= \frac{1}{\sqrt{2\pi\hbar}} \frac{\hbar}{(-ip)} N \frac{d}{dp} \int_0^a e^{-ipx/\hbar} dx$

$= \frac{1}{\sqrt{2\pi\hbar}} \frac{\hbar}{(-ip)} N \frac{d}{dp} \frac{\hbar}{(-ip)} e^{-ipx/\hbar} \Big|_0^a$

$= \frac{-\hbar^2}{\sqrt{2\pi\hbar}\, p} N \frac{d}{dp} \left(\frac{1}{p} \left(e^{-ipa/\hbar} - 1 \right) \right)$

$= \frac{-\hbar^2}{\sqrt{2\pi\hbar}\, p} N \left(-\frac{1}{p^2} \left(e^{-ipa/\hbar} - 1 \right) - \frac{ia}{\hbar p} e^{-ipa/\hbar} \right)$

$\boxed{4}$ a) $e^{-\frac{i}{\hbar}\hat{H}t} = e^{-i\omega\sigma_y t} = e^{-i\omega t(|R\rangle\langle R| - |L\rangle\langle L|)}$

$$= e^{-i\omega t}|R\rangle\langle R| + e^{i\omega t}|L\rangle\langle L|$$

$$= \frac{1}{2}e^{-i\omega t}\begin{pmatrix}1\\ i\end{pmatrix}(1\ -i) + \frac{1}{2}e^{+i\omega t}\begin{pmatrix}1\\ -i\end{pmatrix}(1\ i)$$

$$= \frac{1}{2}\begin{pmatrix} e^{-i\omega t}+e^{i\omega t} & -ie^{-i\omega t}+ie^{i\omega t} \\ ie^{-i\omega t}-ie^{i\omega t} & e^{-i\omega t}+e^{i\omega t}\end{pmatrix}$$

$$= \begin{pmatrix}\cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t\end{pmatrix}$$

b) $|\psi(t)\rangle = e^{-i/\hbar \hat{H}t}|\psi(0)\rangle = \begin{pmatrix}\cos\omega t \\ \sin\omega t\end{pmatrix}$

$$\langle\sigma_z\rangle(t) = \langle\psi(t)|\sigma_z|\psi(t)\rangle = (\cos\omega t\ \ \sin\omega t)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix}$$

$$= \cos^2\omega t - \sin^2\omega t = \cos 2\omega t$$

c) $\hat{\sigma}_z(t) = e^{i/\hbar \hat{H}t}\sigma_z e^{-i/\hbar \hat{H}t}$

$$= \begin{pmatrix}\cos\omega t & +\sin\omega t\\ -\sin\omega t & \cos\omega t\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\omega t & -\sin\omega t\\ \sin\omega t & \cos\omega t\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

$$= \begin{pmatrix}\cos\omega t & \sin\omega t\\ -\sin\omega t & \cos\omega t\end{pmatrix}\begin{pmatrix}\cos\omega t & -\sin\omega t\\ -\sin\omega t & -\cos\omega t\end{pmatrix}$$

$$= \begin{pmatrix}\cos^2\omega t - \sin^2\omega t & -2\cos\omega t\ \sin\omega t\\ -2\cos\omega t\ \sin\omega t & \sin^2\omega t - \cos^2\omega t\end{pmatrix}$$

$$\langle\sigma_z\rangle(t) = \langle\psi(0)|\hat{\sigma}_z(t)|\psi(0)\rangle = \cos^2\omega t - \sin^2\omega t = \cos 2\omega t$$

$\boxed{5}$ a) $N = \frac{1}{\sqrt{2}}$ because $\langle H\, \alpha | V\, -\alpha \rangle = 0$

b) $_A\langle \Theta | \Psi \rangle = \left(\cos\Theta \langle H| + \sin\Theta \langle V|\right)_A \frac{1}{\sqrt{2}} \left(|H\, \alpha\rangle + |V\, -\alpha\rangle \right)$

$= \frac{1}{\sqrt{2}} \left(\cos\Theta\, |\alpha\rangle + \sin\Theta\, |-\alpha\rangle \right)$

$pr_{A|\Theta\rangle} = \frac{1}{2} \left(\cos\Theta \langle\alpha| + \sin\Theta \langle -\alpha| \right) \left(\cos\Theta\, |\alpha\rangle + \sin\Theta\, |-\alpha\rangle \right)$

$= \frac{1}{2} \left(\cos^2\Theta + \sin^2\Theta + 2\cos\Theta \sin\Theta \langle\alpha|-\alpha\rangle \right)$

$= \frac{1}{2} \left(1 + \sin 2\Theta\, e^{-2\alpha^2} \right)$

$\langle \frac{\pi}{2} - \Theta | \Psi \rangle = \left(-\sin\Theta \langle H| + \cos\Theta \langle V| \right) \frac{1}{\sqrt{2}} \left(|H\, \alpha\rangle + |V\, -\alpha\rangle \right)$

$= \frac{1}{\sqrt{2}} \left(-\sin\Theta\, |\alpha\rangle + \cos\Theta\, |-\alpha\rangle \right)$

$pr_{A|\frac{\pi}{2}+\Theta\rangle} = \frac{1}{2} \left(1 - \sin 2\Theta\, e^{-2\alpha^2} \right)$

$\langle n | \pm\alpha \rangle = e^{-\alpha^2/2} (\pm\alpha)^n / \sqrt{n!}$

$_B\langle n | \Psi \rangle = {}_B\langle n | \frac{1}{\sqrt{2}} \left(|H\, \alpha\rangle + |V\, -\alpha\rangle \right)$

$= \frac{1}{\sqrt{2n!}} e^{-\alpha^2/2} \left(\alpha^n |H\rangle + (-\alpha)^n |V\rangle \right)$

$pr_{B,|n\rangle} = \frac{1}{2n!} e^{-\alpha^2} \left(\alpha^{2n} + (-\alpha)^{2n} \right) = \frac{1}{n!} e^{-\alpha^2} \alpha^{2n}$

c) $|\Psi_{123}\rangle = \frac{1}{\sqrt{2}} \left(a|H\rangle + b|V\rangle \right) \left(|H\, \alpha\rangle + |V\, -\alpha\rangle \right)$

$= \frac{1}{\sqrt{2}} \left(a\,|H\,H\,\alpha\rangle + a\,|H\,V\, -\alpha\rangle + b\,|V\,H\,\alpha\rangle + b\,|V\,V\,-\alpha\rangle \right)$

$\langle \Phi_+ | \Psi_{123} \rangle = \frac{1}{2} \left(a|\alpha\rangle + b|-\alpha\rangle \right)$

$pr = \frac{1}{4} \left(aa^* + bb^* + (ab^* + ba^*) \langle\alpha|-\alpha\rangle \right)$

$= \frac{1}{4} \left(1 + 2\,\mathrm{Re}\, ab\; e^{-2\alpha^2} \right)$

$$\langle \Phi_- | \Psi_{123} \rangle = \frac{1}{2}\left(a|\alpha\rangle - b|-\alpha\rangle\right)$$

$$pr = \frac{1}{4}\left(1 - 2\,\mathrm{Re}\; ab\; e^{-2\alpha^2}\right)$$

$$\langle \Psi_+ | \Psi_{123} \rangle = \frac{1}{2}\left(a|-\alpha\rangle + b|\alpha\rangle\right)$$

$$pr = \frac{1}{4}\left(1 + 2\,\mathrm{Re}\; ab\; e^{-2\alpha^2}\right)$$

$$\langle \Psi_- | \Psi_{123} \rangle = \frac{1}{2}\left(a|-\alpha\rangle - b|\alpha\rangle\right)$$

$$pr = \frac{1}{4}\left(1 - 2\,\mathrm{Re}\; ab\; e^{-2\alpha^2}\right)$$