Phys 443 Winter 2015

Midterm examination

Solutions

1

$$[X^2+P^2,\ X+iP] = [P^2\ X] + i[X^2\ P]$$

$$= 2P[P\ X] + 2iX[X\ P] = -2iP - 2X$$

2

$$\langle A \rangle = 1/2$$

$$\langle \Delta A^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2 = 1/4$$

$$\langle B \rangle = 2\tfrac{1}{2}$$

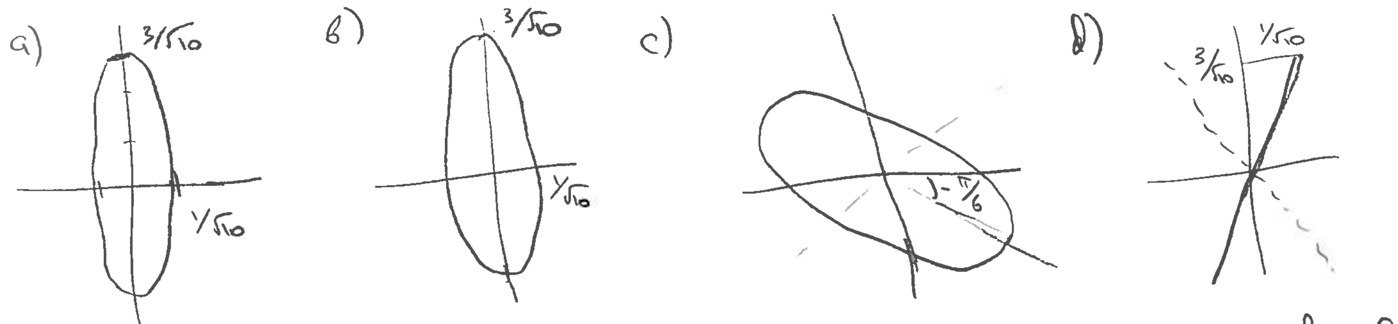
$$\langle \Delta B^2 \rangle = 3/4$$

Uncertainty principle: $\langle \Delta A^2 \rangle \langle \Delta B^2 \rangle \geq \frac{1}{4} \langle |[A\ B]| \rangle^2$

$$\frac{3}{16} \geq \frac{1}{4}|x|^2$$

$$x \leq \sqrt{3}/2$$

3



answer depends on convention

4

a) $|\theta\rangle \to e^{i\varphi}|\theta\rangle$

$|\frac{\pi}{2}+\theta\rangle \to |\frac{\pi}{2}+\theta\rangle$

b) $$\hat{U} = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & 1 \end{pmatrix} = e^{i\varphi}|\theta\rangle\langle\theta| + |\tfrac{\pi}{2}+\theta\rangle\langle\tfrac{\pi}{2}+\theta|$$

c) $$\hat{U} = e^{i\varphi}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}(\cos\theta \quad \sin\theta) + \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}(-\sin\theta \quad \cos\theta)$$

$$= \begin{pmatrix} e^{i\varphi}\cos^2\theta + \sin^2\theta & (e^{i\varphi}-1)\cos\theta\sin\theta \\ (e^{i\varphi}-1)\cos\theta\sin\theta & e^{i\varphi}\sin^2\theta + \cos^2\theta \end{pmatrix}$$

d) $\hat{U} = e^{-i\ldots/\hbar}$

$$H = E_1 |\theta\rangle\langle\theta| + E_2 \left|\tfrac{\pi}{2}+\theta\right\rangle\left\langle\tfrac{\pi}{2}+\theta\right|$$

$$\begin{cases} e^{i\varphi} = e^{-iE_1 t_0/\hbar} \\ 1 = e^{-iE_2 t_0/\hbar} \end{cases} \qquad \Bigg| \qquad E_1 = -\frac{\varphi\hbar}{t_0} \qquad E_2 = 0$$

$\boxed{5}$ a) $|\Psi\rangle = \frac{1}{\sqrt{30}}\left(|HH\rangle + 2|HV\rangle - 3|VH\rangle + 4i|VV\rangle\right)$

b) $\langle +|\Psi\rangle = \frac{1}{\sqrt{60}}\left(|H\rangle + 2|V\rangle - 3|H\rangle + 4i|V\rangle\right) = \frac{1}{\sqrt{60}}\left(-2|H\rangle + (2+4i)|V\rangle\right)$

$pr_+ = \| \langle +|\Psi\rangle \|^2 = \frac{1}{60}(2^2+2^2+4^2) = \frac{2}{5}$

$\langle -|\Psi\rangle = \frac{1}{\sqrt{60}}\left(|H\rangle + 2|V\rangle + 3|H\rangle - 4i|V\rangle\right) = \frac{1}{\sqrt{60}}\left(4|H\rangle + (2-4i)|V\rangle\right)$

$pr_- = \| \langle -|\Psi\rangle \|^2 = \frac{1}{60}(4^2+2^2+4^2) = \frac{3}{5}$

c) $\mathcal{N}\left(\langle +|\Psi\rangle\right) = \frac{1}{\sqrt{24}}\left(-2|H\rangle + (2+4i)|V\rangle\right)$

$\mathcal{N}\left(\langle -|\Psi\rangle\right) = \frac{1}{\sqrt{36}}\left(4|H\rangle + (2-4i)|V\rangle\right)$

$\boxed{6}$ a) $\begin{vmatrix} 1-E & 3i \\ -3i & 9-E \end{vmatrix} = 0$

$(1-E)(9-E) - 9 = 0$

$E^2 - 10E = 0$

$E_1 = 0 \rightarrow$ eigenvector $|E_1\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} -3i \\ 1 \end{pmatrix}$

$E_2 = 10(\hbar\omega) \rightarrow$ eigenvector $|E_2\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ -3i \end{pmatrix}$

b) $pr_{|E_1\rangle} = |\langle \Psi_0 | E_1\rangle|^2 = \left|\frac{1}{\sqrt{20}}(1\ \ -i)\begin{pmatrix} -3i \\ 1 \end{pmatrix}\right|^2 = \frac{4}{5}$

$pr_{|E_2\rangle} = |\langle \Psi_0 | E_2\rangle| = \left|\frac{1}{\sqrt{20}}(1\ \ -i)\begin{pmatrix} 1 \\ -3i \end{pmatrix}\right| = \frac{1}{5}$

$\langle E\rangle = \langle \Psi_0 | \hat{H} | \Psi_0\rangle = \frac{\hbar\omega}{2}(1\ \ -i)\begin{pmatrix} 1 & 3i \\ -3i & 9 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}(1\ \ -i)\begin{pmatrix} -2 \\ 6i \end{pmatrix} = 2\hbar\omega$

$\langle E^2\rangle = \langle \Psi_0 | H^2 | \Psi_0\rangle = \| H|\Psi_0\rangle \|^2 = \left\| \frac{\hbar\omega}{\sqrt{2}}\begin{pmatrix} -2 \\ 6i \end{pmatrix}\right\|^2 = 20(\hbar\omega)^2$

$\langle \Delta E^2\rangle = \langle E^2\rangle - \langle E\rangle^2 = 16(\hbar\omega)^2$

c) $\hat{U} = |E_1\rangle\langle E_1| + e^{-10i\omega t}|E_2\rangle\langle E_2|$

$$= \frac{1}{10}\left[\begin{pmatrix} 9 & -3i \\ 3i & 1 \end{pmatrix} + \begin{pmatrix} 1 & 3i \\ -3i & 9 \end{pmatrix} e^{-10i\omega t}\right]$$

$$\hat{U}|\psi_0\rangle = \frac{1}{10\sqrt{2}}\left[\begin{pmatrix} 9+3 \\ 3i+i \end{pmatrix} + \begin{pmatrix} 1-3 \\ -3i+9i \end{pmatrix} e^{-10i\omega t}\right]$$

$$= \frac{1}{5\sqrt{2}}\begin{bmatrix} 6 - e^{-10i\omega t} \\ 2i + 3i\,e^{-10i\omega t} \end{bmatrix}$$