University of Calgary
Winter semester 2015

PHYS 443: Quantum Mechanics I

Homework assignment 2

Due February 10, 2015

Problem 2.1. Operator $\hat{A}$ has the following matrix in the canonical basis.

$$\hat{A} \simeq \begin{pmatrix} 41 & -12i \\ 12i & 34 \end{pmatrix}$$

a) Present this operator in the form $\hat{A} = v_1 |v_1\rangle\langle v_1| + v_2 |v_2\rangle\langle v_2|$, where $\{|v_1\rangle, |v_2\rangle\}$ is an orthonormal basis. Find v_1, v_2, as well as the matrices of $|v_1\rangle$ and $|v_2\rangle$ in the canonical basis. Write the matrices of outer products $|v_{1,2}\rangle\langle v_{1,2}|$ in the canonical basis and verify explicitly that $\hat{A} = v_1 |v_1\rangle\langle v_1| + v_2 |v_2\rangle\langle v_2|$.

b) Find the matrix $e^{i\hat{A}}$ in the canonical basis. Verify if the result is unitary. Could we predict that the result would be unitary without calculating the exponent?

c) Consider a measurement defined by operator $\hat{A}$ as an observable. We apply this measurement to state $|R\rangle$. What states will this measurement project $|R\rangle$ upon and what are the associated probabilities?

d) Calculate the expectation value of the measurement result

- using the definition of the expectation value from the probability theory;
- using the expression for the quantum mean.

Verify that the results are the same.

e) Calculate the variance of observable $\hat{A}$ in state $|R\rangle$.

f) Calculate the commutator of observables $\hat{\sigma}_x$ and $\hat{A}$

g) Write the uncertainty principle for these observables and state $|R\rangle$. Verify that it holds.

Problem 2.2. Consider operators $\hat{A}_{\lambda/2}$ and $\hat{A}_{\lambda/4}$ associated, respectively, with half- and quarter-wave plates with their optical axes oriented at arbitrary angle θ to horizontal. First, recall (Sec. C.3) that the waveplates perform the following transformations.

$$|\theta\rangle \to -|\theta\rangle \text{ (HWP) or } |\theta\rangle \to i|\theta\rangle \text{ (QWP) ;} \quad (1)$$

$$\left|\frac{\pi}{2} + \theta\right\rangle \to \left|\frac{\pi}{2} + \theta\right\rangle . \quad (2)$$

a) Write operators $\hat{A}_{\lambda/2}$ and $\hat{A}_{\lambda/4}$ in the Dirac notation in terms of $\{|\theta\rangle, \left|\frac{\pi}{2} + \theta\right\rangle\}$[1].

b) Find the matrices of $\hat{A}_{\lambda/2}$ and $\hat{A}_{\lambda/4}$ in the canonical basis by writing the result of part (a) in the matrix form in that basis.

[1] In this case, the overall phase in the right-hand side of Eq. (1) does matter. This is because we are interested not only in the transformation of state $|+\rangle$ itself, but in the whole linear operation this transformation defines. To see the effect of the overall phase, you may want to try solving part (a) using $|\theta\rangle \to |\theta\rangle$ instead of Eq. (1).

c) Express these results in the Dirac notation in terms of outer products of states $|H\rangle$ and $|V\rangle$;

d) Find the matrices of $\hat{A}_{\lambda/2}$ and $\hat{A}_{\lambda/4}$ in basis $\{|\theta\rangle, \left|\frac{\pi}{2}+\theta\right\rangle\}$.

e) Determine the matrices of $\hat{A}_{\lambda/2}$ and $\hat{A}_{\lambda/4}$ in the canonical basis from those in basis $\{|\theta\rangle, \left|\frac{\pi}{2}+\theta\right\rangle\}$ using the method of "inserting $\hat{\mathbf{1}}$" [Eq. (A.27) in the lecture notes]. Is your result consistent with part (b)?

f) Determine the matrices of $\hat{A}^{-1}_{\lambda/2}$ and $\hat{A}^{-1}_{\lambda/4}$ in the canonical basis.
Hint: This part has a very simple solution.

g) Suppose that the waveplate is placed in front of a polarizing beam splitter and used for measuring the polarization state of the photon. Find the measurement bases both for the half- and quarter-wave plate cases.
Hint: What states will $\hat{A}_{\lambda/2}$ and $\hat{A}_{\lambda/4}$ convert into $|H\rangle$ and $|V\rangle$?

h) In the above measurement, the event in which the photon is registered by the detector in the transmitted channel is assigned value $+1$, and in the reflected channel -1. Find the matrix of the corresponding observables in the canonical basis. Verify that this observable becomes $\hat{\sigma}_x$ for the half-wave plate and $\theta = 22\frac{1}{2}^\circ$, and $\hat{\sigma}_y$ for the quarter-wave plate and $\theta = 45^\circ$ (see Ex. 1.25).

i) Suppose operators $\hat{A}_{\lambda/2}$ and $\hat{A}_{\lambda/4}$ correspond to the evolution under some Hamiltonians $\hat{H}_{\lambda/2}$ and $\hat{H}_{\lambda/4}$ for time t_0. Find the matrices of these Hamiltonians in the canonical basis.

j) Using the above result, write the Schrödinger equation for the half-wave plate at $\theta = 30^\circ$. Solve that equation using the matrix method [Eq. (1.25) from the lecture notes] for the initial state $\hat{H}$. Is the result for $t = t_0$ consistent with what you would expect from the physics of polarization transformations?

Use notation $c = \cos\theta,\ s = \sin\theta$.