University of Calgary
Winter semester 2015

PHYS 443: Quantum Mechanics I

Homework assignment 1

Due January 27, 2015

Problem 1.1.

a) Using the applet at `http://www.amanogawa.com/archive/Polarization/Polarization-2.html`, generate states $|+30°\rangle$, $|-60°\rangle$, $|R\rangle$, $|L\rangle$. Record the values of amplitudes and phases used in each case. Verify that these values are consistent with Section C.1 of the lecture notes (note that the applet's definitions of the left and right circular polarizations are opposite to ours).

b) Download the demonstration at `http://demonstrations.wolfram.com/PolarizationOfAnOpticalWaveThroughPolarizersAndWavePlates/` (to run the demo, if you don't have Mathematica, you will also need the Mathematica plugin for your browser or the Wolfram CDF Player available at `http://www.wolfram.com/products/player/download.cgi`). Verify that the transformations of the waves under the action of waveplates is consistent with that described in Section C.3 of the lecture notes. No written reporting is required for part (b).

Problem 1.2. Two states are decomposed in the circular basis according to

$$|\psi\rangle = \frac{1}{\sqrt{5}}(2\,|R\rangle + i\,|L\rangle), \quad |\phi\rangle = \frac{1}{\sqrt{5}}(i\,|R\rangle + 2\,|L\rangle), \tag{1}$$

a) Show that these states form an orthonormal basis using the fact that the circular basis is orthonormal.

b) Find the decompositions of these states in the canonical basis using two methods:

- by expressing $|R\rangle$ and $|L\rangle$ in the canonical basis and substituting into Eq. (1);
- by finding the matrices of $|\psi\rangle$, $|\phi\rangle$, $|H\rangle$ and $|V\rangle$ in the circular basis and using the inner product.

c) Verify that states $|\psi\rangle$ and $|\phi\rangle$ form an orthonormal set using the inner product in the canonical basis.

d) Decompose states $|H\rangle, |V\rangle, |R\rangle, |L\rangle, (|H\rangle + 2i\,|V\rangle)/\sqrt{5}$ in basis $\{|\psi\rangle, |\phi\rangle\}$. Write your answer both in the Dirac and matrix notations.

e) States $|H\rangle, |V\rangle, |R\rangle, |L\rangle, (|H\rangle + 2i\,|V\rangle)/\sqrt{5}$ are measured in basis $\{|\psi\rangle, |\phi\rangle\}$. What are the probabilities of the outcomes?

Problem 1.3. Verify that the following three vectors: $\vec{w}_1 = (0, 2, 1)$, $\vec{w}_2 = (0, 2, 0)$, $\vec{w}_3 = (-1, -1, 0)$ form a basis in the three-dimensional geometrical space. Perform the Gram-Schmidt procedure for this set. Verify that the basis obtained in this procedure is indeed orthonormal.

Problem 1.4. Find the matrices of the operators $\hat{A}_{\lambda/2}$ and $\hat{A}_{\lambda/4}$ associated, respectively, with half- and quarter-wave plates with their optical axes oriented at arbitrary angle θ to horizontal. To that end, recall (Sec. C.3) that the waveplates perform the following transformations.

$$|\theta\rangle \to -\,|\theta\rangle \ \text{(HWP) or} \ |\theta\rangle \to i\,|\theta\rangle \ \text{(QWP)}\ ; \tag{2}$$

$$\left|\frac{\pi}{2} + \theta\right\rangle \to \left|\frac{\pi}{2} + \theta\right\rangle. \tag{3}$$

a) Write the matrices of $\hat{A}_{\lambda/2}$ and $\hat{A}_{\lambda/4}$ in basis $\{|\theta\rangle, \left|\frac{\pi}{2}+\theta\right\rangle\}$[1].

b) Decompose the canonical basis elements into basis $\{|\theta\rangle, \left|\frac{\pi}{2}+\theta\right\rangle\}$.

c) Using the fact that, for any linear operator, $\hat{A}(\lambda|a\rangle+\mu|b\rangle) = \lambda\hat{A}|a\rangle+\mu\hat{A}|b\rangle$, determine from Eqs. (2) and (3) how the vertical and horizontal polarization states are mapped by the waveplate operators.

d) Based on the result of the previous part, write the matrices of $\hat{A}_{\lambda/2}$ and $\hat{A}_{\lambda/4}$ in the canonical basis.

e) Express these results in the Dirac notation in terms of outer products of states $|H\rangle$ and $|V\rangle$;

f) Determine the matrices of $\hat{A}_{\lambda/2}$ and $\hat{A}_{\lambda/4}$ in the canonical basis from those in basis $\{|\theta\rangle, \left|\frac{\pi}{2}+\theta\right\rangle\}$ using the method of "inserting $\hat{\mathbf{1}}$" [Eq. (A.27) in the lecture notes]. Is your result consistent with part (d)?

Use notation $c=\cos\theta,\ s=\sin\theta$.

Problem 1.5 Consider the modified BB84 protocol in which Alice sends and Bob analyzes the photon in a polarization basis that is randomly chosen, with the same probability for each choice, among the following three: $(0°, 90°)$, $(30°, 120°)$, $(60°, 150°)$. Find the bit error rate that Alice and Bob will see in the event of a straightforward "intercept-resend" attack, i.e. if Eve intercepts the photon, measures it in one of the above three bases (randomly chosen with equal probabilities), and resends whatever she detected. There are no losses, all equipment is perfect.

[1] In this case, the overall phase in the right-hand side of Eq. (2) does matter. This is because we are interested not only in the transformation of state $|+\rangle$ itself, but in the whole linear operation this transformation defines. To see the effect of the overall phase, you may want to try solving part (a) using $|\theta\rangle \to |\theta\rangle$ instead of Eq. (2).