Phys 443 Winter 2015

Final examination

Solutions

$\boxed{1}$

$$
\begin{aligned}
[L_x L_y, L_z^2] &= L_x [L_y, L_z^2] + [L_x, L_z^2] L_y \\
&= L_x [L_y, L_z] L_z + L_x L_z [L_y, L_z] \\
&\quad + [L_x, L_z] L_z L_y + L_z [L_x, L_z] L_y \\
&= i\hbar \left(L_x^2 L_z + L_x L_z L_x - L_y L_z L_y - L_z L_y^2 \right) \quad *
\end{aligned}
$$

$\boxed{2}$ a) $\hat{H} = \hbar\omega\left(\hat{n} + \frac{1}{2}\right) \Rightarrow E_n = \hbar\omega\left(n + \frac{1}{2}\right)$, where $\omega = \sqrt{\frac{k}{m}}$

$$
|n\rangle = \int \frac{H_n(X)}{\pi^{1/4}\sqrt{2^n n!}} e^{-X^2/2} |X\rangle dX \qquad \Bigg| \quad X = x\sqrt{\frac{m\omega}{\hbar}}, \quad |X\rangle = \left(\frac{\hbar}{m\omega}\right)^{1/4} |x\rangle
$$

$$
= \int \underbrace{\frac{H_n\left(x\sqrt{\frac{m\omega}{\hbar}}\right)}{\pi^{1/4}\sqrt{2^n n!}} e^{-\frac{m\omega x^2}{2\hbar}} \left(\frac{m\omega}{\hbar}\right)^{1/4}}_{\psi_n(x)} |x\rangle dx
$$

b) Each of the above wavefunctions satisfies the time-independent Schrödinger equation for the new potential in region $x<0$, but only odd ones satisfy the boundary requirement $\psi(0) = 0$. Hence the new energy eigenvalues are

$$
E'_j = E_{n=2j+1} = \hbar\omega\left(2j + \frac{3}{2}\right), \quad j = 0, 1, \dots
$$

$$
\psi'_j(x) = \mathcal{N} \begin{cases} \psi_{n=2j+1}(x), & x<0 \\ 0, & x>0 \end{cases}
= \begin{cases} \sqrt{2}\,\psi_{n=2j+1}(x), & x<0 \\ 0, & x>0 \end{cases}
$$

Alternative answer to problem 1:

$$
i\hbar \left[L_x^2 L_z + \left(L_z L_x L_x - \cancel{i\hbar L_y L_x} \right) - \left(L_y L_y L_z - \cancel{i\hbar L_y L_x} \right) - L_z L_y^2 \right]
$$

$$
= i\hbar \left[L_x^2 L_z + L_z L_x^2 - L_y^2 L_z - L_z L_y^2 \right]
$$

$\boxed{3}$ a) $\langle + | \Psi \rangle = \frac{1}{\sqrt{2}\sqrt{6}} \left(\langle H| + \langle V| \right)\left([|H\rangle + |V\rangle] \otimes |1, 1\rangle + 2|V\rangle \otimes |1, -1\rangle \right)$

$$= \frac{1}{\sqrt{12}} \left(2|1, 1\rangle + 2|1, -1\rangle \right) = \frac{1}{\sqrt{3}} \left(|1, 1\rangle + |1, -1\rangle \right)$$

$$Pr_{Alice,+} = \| \langle + | \Psi \rangle \|^2 = \frac{2}{3}$$

$$\langle - | \Psi \rangle = \frac{1}{\sqrt{2}\sqrt{6}} \left(\langle H| - \langle V| \right)\left([|H\rangle + |V\rangle] \otimes |1, 1\rangle + 2|V\rangle \otimes |1, -1\rangle \right)$$

$$= -\frac{1}{\sqrt{3}} |1, -1\rangle$$

$$Pr_{Alice,-} = \frac{1}{3}$$

b) $L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\qquad L_x^2 = \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$$\mathcal{N}\left(\langle + | \Psi \rangle \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \equiv |\varphi_+\rangle$$

$$\langle \varphi_+ | L_x | \varphi_+ \rangle = \frac{\hbar}{2\sqrt{2}} (1\ 0\ 1) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{\hbar}{2\sqrt{2}} (1\ 0\ 1) \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 0$$

$$\langle \varphi_+ | L_x^2 | \varphi_+ \rangle = \frac{\hbar^2}{4} (1\ 0\ 1) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{\hbar^2}{4} (1\ 0\ 1) \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = \hbar^2 \quad \Rightarrow \sqrt{\langle \Delta L_x^2 \rangle} = \hbar$$

$$\mathcal{N}\left(\langle - | \Psi \rangle \right) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \equiv |\varphi_-\rangle$$

$$\langle \varphi_- | L_x | \varphi_- \rangle = \frac{\hbar}{\sqrt{2}} (0\ 0\ 1) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{\hbar}{\sqrt{2}} (0\ 0\ 1) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\langle \varphi_- | L_x^2 | \varphi_- \rangle = \frac{\hbar^2}{2} (0\ 0\ 1) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{\hbar^2}{2} (0\ 0\ 1) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{\hbar^2}{2} \quad \Rightarrow \sqrt{\langle \Delta L_x^2 \rangle} = \frac{\hbar}{\sqrt{2}}$$

c) $\langle L_x \rangle = pr_+ \langle \varphi_+ | L_x | \varphi_+ \rangle + pr_- \langle \varphi_- | L_x | \varphi_- \rangle = 0$

$$\langle L_x^2 \rangle = pr_+ \langle \varphi_+ | L_x^2 | \varphi_+ \rangle + pr_- \langle \varphi_- | L_x^2 | \varphi_- \rangle$$

$$= \frac{2}{3} \hbar^2 + \frac{1}{3} \frac{\hbar^2}{2} = \frac{5}{6} \hbar^2$$

$$\sqrt{\langle \Delta L_x^2 \rangle} = \sqrt{\frac{5}{6}}\, \hbar$$

4 a)

$$\psi(x) = \begin{cases} 0, & x<0 \\ Ae^{ikx} + Be^{-ikx}, & 0<x<a \\ Ce^{+\varkappa_0 x} + De^{-\varkappa_0 x}, & a<x<b \\ Ee^{-\varkappa_1 x}, & x>b \end{cases}$$

where $k = \dfrac{\sqrt{2mE}}{\hbar}$, $\varkappa_0 = \dfrac{\sqrt{2m(V_0-E)}}{\hbar}$, $\varkappa_1 = \dfrac{\sqrt{2m(V_1-E)}}{\hbar}$

b) Boundary conditions:

$\psi(0)=0 \Rightarrow A=-B$

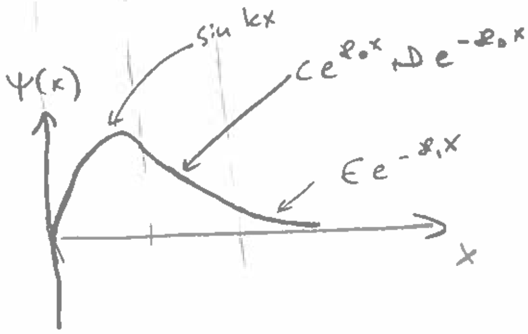
$\psi(a+0) = \psi(a-0) \Rightarrow A(e^{ika} - e^{-ika}) = Ce^{+\varkappa_0 a} + De^{-\varkappa_0}$

$\psi'(a+0) = \psi'(a-0) \Rightarrow ikA(e^{ika} + e^{-ika}) = \varkappa_0(Ce^{\varkappa_0 a} - De^{-\varkappa_0}$

$\psi(b+0) = \psi(b-0) \Rightarrow Ce^{\varkappa_0 b} + De^{-\varkappa_0 b} = Ee^{-\varkappa_1 b}$

$\psi'(b+0) = \psi'(b-0) \Rightarrow \varkappa_0(Ce^{\varkappa_0 b} - De^{-\varkappa_0 b}) = -E\varkappa_1 e^{-\varkappa_1}$

c)



$\boxed{5}$

a) $\langle L^2 \rangle = \hbar^2 \ell(\ell+1) = 2\hbar^2$

b) $\hat{L}_x |\psi(0)\rangle = \hbar |\psi(0)\rangle$

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \hbar \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} y \\ x+z \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$y = x\sqrt{2} = z\sqrt{2}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathcal{N} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = |\psi(0)\rangle$$

c) $\psi_0(\theta, \varphi) = \frac{1}{2} \left[Y_1^1(\theta,\varphi) + \sqrt{2} Y_1^0(\theta, \varphi) + Y_1^{-1}(\theta, \varphi) \right]$

$$= \frac{1}{2} \sqrt{\frac{3}{8\pi}} \left[-\sin\theta\, e^{i\varphi} + 2\cos\theta + \sin\theta\, e^{-i\varphi} \right]$$

d) $H = \hbar\omega \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

e) Eigenvalues and eigenstates

$v_1 = 0 \qquad |v_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$v_2 = \hbar\omega \qquad |v_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$v_3 = -\hbar\omega \qquad |v_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$$\hat{U}(t) = e^{-i\frac{\hat{H}}{\hbar}t} = |v_1\rangle\langle v_1| + e^{-i\omega t}|v_2\rangle\langle v_2| + e^{i\omega t}|v_3\rangle\langle v_3|$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2} e^{-i\omega t} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} e^{+i\omega t} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\omega t & -i\sin\omega t & 0 \\ -i\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

f) $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle = \frac{1}{2} \begin{pmatrix} \cos\omega t - i\sqrt{2}\sin\omega t \\ \sqrt{2}\cos\omega t - i\sin\omega t \\ 1 \end{pmatrix}$