University of Calgary
Winter semester 2011

PHYS 443: Quantum Mechanics I

Homework assignment 2

Due February 8, 2011

Problem 2.1. As discussed in class, quantum cryptography becomes insecure when the measurement error rate due to dark counts of Bob's single photon detectors exceeds ∼25% of all detection events. Assuming that Alice has a perfect single photon source, estimate the maximum possible secure communication distance and the bit transfer rate at this distance given the following parameters:

- photon loss in the fiber communication line: 5%/km;
- emission rate of Alice's source: 10^6 photons per second;
- quantum efficiency of the photon detectors (i.e. the probability that the detector will "click" when hit by a single photon): 10%;
- probability for each detector to produce a dark count simultaneously with the photon pulse: 10^{-5} per pulse.

Problem 2.2. Consider an operator $\hat{A}$ that performs the following transformation.

$$|H\rangle \quad \to \quad \frac{2\,|H\rangle + i\,|V\rangle}{\sqrt{5}}; \tag{1}$$

$$|+\rangle \quad \to \quad \frac{2+i}{\sqrt{5}}\,|+\rangle\,. \tag{2}$$

a) How is the vertical polarization state mapped by $\hat{A}$?[1]

b) Write the matrix of $\hat{A}$ in the canonical basis.

c) Using the fact that, for any linear operator, $\hat{A}(\lambda\,|a\rangle + \mu\,|b\rangle) = \lambda\hat{A}\,|a\rangle + \mu\hat{A}\,|b\rangle$, determine how $\hat{A}$ acts upon the circular polarization states.

d) Using the previous result, find the matrix of $\hat{A}$ in the circular polarization basis;

e) Find the matrix of $\hat{A}$ in the canonical basis from its matrix in the circular basis using the method of "inserting $\hat{\mathbf{1}}$" (Note 1.20 in the *printed* lecture notes). Is your result consistent with that of part (b)?

f) Find the traces of the matrices of $\hat{A}$ in the canonical and circular bases. Are they identical?

g) Express $\hat{A}$ in the Dirac notation in terms of outer products of states $|H\rangle$ and $|V\rangle$;

h) Is $\hat{A}$ Hermitean? If not, what is its adjoint?

[1] In this case, the overall phase in the right-hand side of Eq. (2) does matter. This is because we are interested not only in the transformation of state $|+\rangle$ itself, but in the whole linear operation this transformation defines. To see the effect of the overall phase, you may want to try solving part (a) using $|+\rangle \to |+\rangle$ instead of Eq. (2).

Problem 2.3. Consider an apparatus for measuring the photon polarization that has the following properties:

- whenever a linearly polarized photon at angle θ enters the apparatus, it displays "2";
- whenever a linearly polarized photon at angle $\pi/2+\theta$ enters the apparatus, it displays "3";
- for photons with polarizations other than the above, it randomly displays one of these numbers with some probabilities.

a) Find the eigenvalues and the eigenstates of the operator $\hat{A}$ associated with the observable measured by this apparatus (**Hint:** you need not solve any equations).

b) Find the matrices of $\hat{A}$ in its eigenbasis and in the $\{|H\rangle, |V\rangle\}$ basis.

c) Find the probability of each measurement outcome for a linearly polarized photon at angle φ.

d) Find the expectation value of this measurement using (i) the result of part (c) and the classical definition

$$\langle Q\rangle = \sum_{i=1}^{N} \mathrm{pr}_i Q_i$$

and (ii) quantum-mechanical formula

$$\langle X\rangle = \left\langle \psi \middle| \hat{X} \middle| \psi \right\rangle .$$

Verify that these results are identical.

Problem 2.4. An atom has four energy levels, $|E_0\rangle$, $|E_1\rangle$, $|E_2\rangle$ and $|E_3\rangle$ with energies $E_0=0$, $E_1=\hbar\omega$, $E_2=2\hbar\omega$, $E_3=3\hbar\omega$, respectively. The atom is in the state $|\psi_0\rangle = N(|E_1\rangle - 4i\,|E_2\rangle + 2\,|E_3\rangle)$, where N is the normalization factor.

a) Write the operator corresponding to a measurement of the atom's energy.

b) Find N.

c) What is the probability to detect the atom in energy eigenstate $|E_1\rangle$?

d) What is the expectation value of the energy measured in the state $|\psi_0\rangle$?