Phys 443

Final examination

Solutions

1

a) $\psi_\alpha(x) = \langle x | \alpha \rangle = \frac{1}{\pi^{1/4}} e^{-(x-\alpha\sqrt{2})^2/2}$

$\psi(x) = \langle x | \psi \rangle = \langle x | P | \psi \rangle = -i \frac{d}{dx} \psi_\alpha(x) = \frac{-i}{\pi^{1/4}} (x - \alpha\sqrt{2}) e^{-(x-\alpha\sqrt{2})^2/2}$

b) $|\alpha\rangle = \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} \alpha^n / \sqrt{n!}\, |n\rangle$

$\hat{P}|\alpha\rangle = \frac{1}{\sqrt{2}i} (\hat{a} - \hat{a}^+) |\alpha\rangle = \frac{e^{-|\alpha|^2/2}}{\sqrt{2}i} \left(\sum_{n=0}^{\infty} \frac{\alpha^{n+1}}{\sqrt{n!}} |n\rangle + \sum_{n=0}^{\infty} \frac{\alpha^n \sqrt{n+1}}{\sqrt{n!}} |n+1\rangle \right)$

2

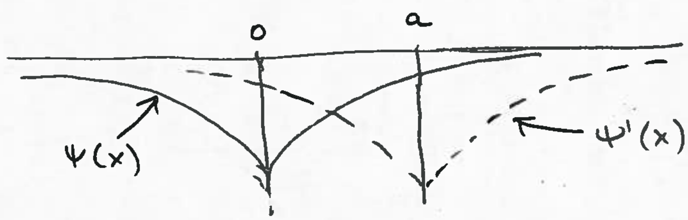
$|\Psi\rangle = (3|+-+\rangle + 4|-+-\rangle)/5$

$\langle H_A | \Psi \rangle = \left(\frac{3}{\sqrt{2}} |-+\rangle + \frac{4}{\sqrt{2}} |+-\rangle \right)/5$

$|\psi_c\rangle = \langle H_A V_B | \Psi \rangle = \left(-\frac{3}{2} |+\rangle + \frac{4}{2} |-\rangle \right)/5 = -\frac{1}{10} (3|+\rangle - 4|-\rangle)$

Probability: $\langle \psi_c | \psi_c \rangle = 1/4$

3



Old bound state $\quad \psi(x) = \sqrt{k_0}\, e^{-k_0|x|}$

New bound state $\quad \psi'(x) = \sqrt{k_0}\, e^{-k_0|x-a|}$

Probability that the state will remain bound

$$|\langle \psi | \psi' \rangle|^2 = \left| \int \psi(x) \psi'(x)\, dx \right|^2$$

$$= k_0 \left(\int_{-\infty}^{0} e^{2k_0 x - k_0 a} dx + \int_0^a e^{-k_0 a} dx + \int_a^{\infty} e^{-2k_0 x + k_0 a} dx \right)^2$$

$$= k_0^2 \left(\frac{e^{-k_0 a}}{2k_0} + a e^{-k_0 a} + \frac{e^{-k_0 a}}{2k_0} \right)^2$$

$$= e^{-2k_0 a} (1 + k_0 a)^2$$

Probability that the particle leaves the bound state

$$1 - |\langle \psi | \psi' \rangle|^2 = 1 - e^{-2k_0 a}(1 + k_0 a)^2$$

$\boxed{4}$ a) $\psi(x, t=0) = \frac{1}{\pi^{1/4}}\left(\alpha - \beta \frac{2x^2 - 1}{\sqrt{2}}\right) e^{-x^2/2}$

b) $|\psi(t)\rangle = e^{-i\omega t/2}\left(\alpha|0\rangle - \beta|2\rangle e^{-2i\omega t}\right)$

c) $\langle E \rangle = \langle \psi(t) | \hbar\omega(\hat{n} + 1/2) | \psi(t) \rangle$

$$= \hbar\omega\left(\frac{\alpha^2}{2} + \frac{5\beta^2}{2}\right)$$

$$\langle E^2 \rangle = \langle \psi(t) | \hbar^2\omega^2(\hat{n} + \tfrac{1}{2})^2 | \psi(t) \rangle$$

$$= \hbar^2\omega^2\left(\frac{\alpha^4}{4} + \frac{25\beta^4}{4}\right)$$

$$\langle \Delta E^2 \rangle = \hbar^2\omega^2\left(\frac{\alpha^2}{4} + \frac{25\beta^2}{4} - \frac{\alpha^4}{4} - \frac{5\alpha^2\beta^2}{2} - \frac{25\beta^4}{4}\right)$$

d) $\langle X \rangle = \langle \psi(t) | \frac{a + a^+}{\sqrt{2}} | \psi(t) \rangle = 0$

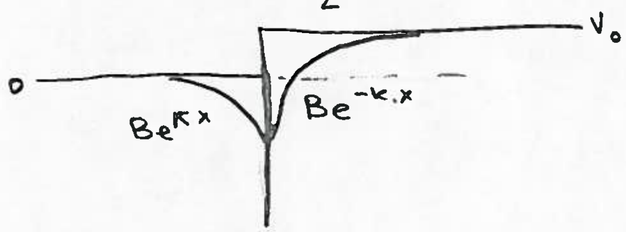
$$\langle \Delta X^2 \rangle = \langle X^2 \rangle =$$

$$= \left(\alpha\langle 0| - \beta\langle 2| e^{2i\omega t}\right) \frac{a^2 + aa^+ + a^+a + a^{+2}}{2} \left(\alpha|0\rangle - \beta|2\rangle e^{-2i\omega t}\right)$$

$$= \frac{1}{2}\alpha^2 + \frac{5}{2}\beta^2 - \frac{\alpha\beta\sqrt{2}}{2}\left(e^{2i\omega t} + e^{-2i\omega t}\right)$$

$$= \frac{1}{2} + 2\beta^2 - \alpha\beta\sqrt{2}\cos\omega t$$

$\boxed{5}$



$$\kappa = \frac{\sqrt{-2mE}}{\hbar}$$

$$\kappa_1 = \frac{\sqrt{-2mE + 2mV_0}}{\hbar} = \sqrt{\kappa^2 + k_0^2} \quad \text{where} \quad k_0 = \frac{\sqrt{2mV_0}}{\hbar}$$

Schrödinger equation at $x = 0$

$$-\int_{-0}^{+0} \frac{\hbar^2}{2m} \frac{d^2}{dx^2} \Psi(x) = \int_{-0}^{+0} (E - V(x)) \Psi(x)$$

$$-\frac{\hbar^2}{2m}\left(-Bk_1 - Bk\right) = W_0 B$$

$$k + \sqrt{k^2 + k_0^2} = \frac{2mW_0}{\hbar^2}$$

$$k^2 + k_0^2 = \left(\frac{2mW_0}{\hbar^2} - k\right)^2$$

$$k_0^2 = \left(\frac{2mW_0}{\hbar^2}\right)^2 - \frac{4mW_0 k}{\hbar^2}$$

$$k = \frac{\left(\frac{2mW_0}{\hbar^2}\right)^2 - k_0^2}{\frac{4mW_0}{\hbar^2}} = \frac{mW_0}{\hbar^2} - \frac{k_0^2 \hbar^2}{4mW_0} = \frac{mW_0}{\hbar^2} - \frac{V_0}{2W_0}$$

$$E = -\frac{(\hbar k)^2}{2m}$$

Only one bound state. Exists if $k > 0$

$$\frac{mW_0}{\hbar^2} > k_0^2 \frac{\hbar^2}{4mW_0}$$

$$\left(\frac{2mW_0}{\hbar^2}\right)^2 > \frac{2mV_0}{\hbar^2}$$

$$\frac{2mW_0^2}{\hbar^2} > V_0$$