PHYS 443

First Midterm Examination

Solutions

1

a) $pr_2 = \left|\frac{2i}{\sqrt{14}}\right|^2 = \frac{2}{7}$

b) $\langle\psi_0|\hat{H}|\psi_0\rangle = (E_1\langle E_1|E_1\rangle + 4E_2\langle E_2|E_2\rangle + 9E_3\langle E_3|E_3\rangle)/14$

$= (4\hbar\omega + 36\hbar\omega)/14 = \frac{20}{7}\hbar\omega$

c) $|\psi_t\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi_0\rangle = \left(e^0|E_1\rangle\langle E_1| + e^{-i\frac{\pi}{2}}|E_2\rangle\langle E_2| + e^{-2i\pi}|E_3\rangle\langle E_3|\right)|\psi_0\rangle$

$= (|E_1\rangle + (-i)(-2i)|E_2\rangle + 3|E_3\rangle)/\sqrt{14}$

$= (|E_1\rangle - 2|E_2\rangle + 3|E_3\rangle)/\sqrt{14}$

$pr = |\langle\psi_0|\psi_t\rangle|^2 = \frac{1}{14^2}|1-4i+9|^2 = \frac{10^2+4^2}{196} = \frac{29}{49}$

2 $\hat{A} = 3|H\rangle\langle H| + \sqrt{3}i|H\rangle\langle V| - \sqrt{3}i|V\rangle\langle H| + |V\rangle\langle V| \leftrightarrow \begin{pmatrix} 3 & \sqrt{3}i \\ -\sqrt{3}i & 1 \end{pmatrix}$

Characteristic equation

$(3-\lambda)(1-\lambda) - (-\sqrt{3}i)(\sqrt{3}i) = 0$

$\lambda^2 - 4\lambda = 0$

$\lambda_1 = 0 \quad \lambda_2 = 4$

Eigenvectors

$\underline{\lambda_1}$ $\begin{pmatrix} 3 & \sqrt{3}i \\ -\sqrt{3}i & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} a \\ b \end{pmatrix} = \mathcal{N}\begin{pmatrix} 1 \\ \sqrt{3}i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3}i \end{pmatrix} \equiv |\psi_1\rangle$

$\underline{\lambda_2}$ $\begin{pmatrix} 3-4 & \sqrt{3}i \\ -\sqrt{3}i & 1-4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} a \\ b \end{pmatrix} = \mathcal{N}\begin{pmatrix} \sqrt{3}i \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{3}i \\ 1 \end{pmatrix} \equiv |\psi_2\rangle$

Check!

$\hat{A} = 0|\psi_1\rangle\langle\psi_1| + 4|\psi_2\rangle\langle\psi_2| = 4\,\frac{1}{4}\begin{pmatrix} \sqrt{3}i \\ 1 \end{pmatrix}(-\sqrt{3}i \quad 1) = \begin{pmatrix} 3 & \sqrt{3}i \\ -\sqrt{3}i & 1 \end{pmatrix} \checkmark$

$e^{i\frac{\pi}{4}\hat{A}} = \underbrace{e^0}_{1}|\psi_1\rangle\langle\psi_1| + \underbrace{e^{i\pi}}_{-1}|\psi_2\rangle\langle\psi_2| \leftrightarrow \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{3}i \\ \sqrt{3}i & 3 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 3 & \sqrt{3}i \\ \sqrt{3}i & 1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3}i \\ \sqrt{3}i & 1 \end{pmatrix}$

$\boxed{3}$

a) $|2\theta\rangle \leftrightarrow \begin{pmatrix} \cos 2\theta \\ \sin 2\theta \end{pmatrix}$ eigenvalue 1

$|\frac{\pi}{2}+2\theta\rangle \leftrightarrow \begin{pmatrix} -\sin 2\theta \\ \cos 2\theta \end{pmatrix}$ eigenvalue 2

b) $\hat{A} = 1\,|2\theta\rangle\langle 2\theta| + 2\,|\frac{\pi}{2}+2\theta\rangle\langle\frac{\pi}{2}+2\theta|$

$$\leftrightarrow \begin{pmatrix} \cos^2 2\theta + 2\sin^2 2\theta & -\cos 2\theta \sin 2\theta \\ -\cos 2\theta \sin 2\theta & \sin^2 2\theta + 2\cos^2 2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \sin^2 2\theta & -\cos 2\theta \sin 2\theta \\ -\cos 2\theta \sin 2\theta & 1 + \cos^2 2\theta \end{pmatrix}$$