PHYS 443

Final Examination

Solutions

$\boxed{1}$ a) As we know from class, the eigenstates of $\hat{\sigma}_x$ are

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

Hence

$$|\psi\rangle = \frac{1}{\sqrt{5}}|+\rangle + \frac{2}{\sqrt{5}}|-\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix}3\\-1\end{pmatrix}$$

b) The probabilities are $\left(\frac{3}{\sqrt{10}}\right)^2 = \frac{9}{10}$ and $\left(\frac{1}{\sqrt{10}}\right)^2 = \frac{1}{10}$

$\boxed{2}$ a) In some basis $\{|v_i\rangle\}$,

$$\mathrm{Tr}\,(\hat{A}\hat{B}) = \sum_i \langle v_i|\hat{A}\underset{\hat{\mathbb{1}}}{\,}\hat{B}|v_i\rangle = \sum_i\sum_j \langle v_i|\hat{A}|v_j\rangle\langle v_j|\hat{B}|v_i\rangle$$

On the other hand,

$$\mathrm{Tr}\,(\hat{B}\hat{A}) = \sum_i \langle v_i|\hat{B}\hat{A}|v_i\rangle = \sum_i\sum_j \langle v_i|B|v_j\rangle\langle v_j|\hat{B}|v_i\rangle.$$

Exchanging the dummy indices i and j, we find

$$\mathrm{Tr}\,(\hat{A}\hat{B}) = \mathrm{Tr}\,(\hat{B}\hat{A})$$

b) $\mathrm{Tr}\,[\hat{A},\hat{B}] = \mathrm{Tr}\,(\hat{A}\hat{B} - \hat{B}\hat{A}) = \mathrm{Tr}\,(\hat{A}\hat{B}) - \mathrm{Tr}\,(\hat{B}\hat{A}) = 0$

On the other hand,

$\mathrm{Tr}\,(i\hbar\,\hat{\mathbb{1}}) = i\hbar N,$

where N is the dimension of the Hilbert space.

$\boxed{3}$ H is a Hermitian operator $\Rightarrow \hat{H}^{\dagger} = \hat{H}$

$\Rightarrow a, d$ are real numbers;

$b^* = c$

$\boxed{4}$ a) We could solve the problem using the characteristic equation, but it is easier to notice that

$$\hat{H} = h\hat{\mathbb{1}} + g\hat{\sigma}_x$$

Because any state is an eigenstate of $\hat{\mathbb{1}}$, the eigenstates of $\hat{\sigma}_x$, $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ are also eigenstates of $\hat{H}$, with eigenvalues $h+g$ and $h-g$.

b) $\hat{H} = (h+g)|+\rangle\langle+| + (h-g)|-\rangle\langle-|$

$$e^{-\frac{i}{\hbar}\hat{H}t} = e^{-i\frac{h+g}{\hbar}t}|+\rangle\langle+| + e^{-i\frac{h-g}{\hbar}t}|-\rangle\langle-|$$

$$= \frac{1}{2}\left[e^{-i\frac{\hbar+g}{\hbar}t}\begin{pmatrix}1\\1\end{pmatrix}(1\ \ 1) + e^{-i\frac{h-g}{\hbar}t}\begin{pmatrix}1\\-1\end{pmatrix}(1\ \ -1)\right]$$

$$= \frac{1}{2}e^{-i\frac{h}{\hbar}t}\begin{bmatrix} e^{-i\frac{g}{\hbar}t} + e^{i\frac{g}{\hbar}t} & e^{-i\frac{g}{\hbar}t} - e^{i\frac{g}{\hbar}t} \\ e^{-i\frac{g}{\hbar}t} - e^{i\frac{g}{\hbar}t} & e^{-i\frac{g}{\hbar}t} + e^{i\frac{g}{\hbar}t} \end{bmatrix}$$

$$= e^{-i\frac{h}{\hbar}t}\begin{bmatrix} \cos\frac{gt}{\hbar} & -i\sin\frac{gt}{\hbar} \\ -i\sin\frac{gt}{\hbar} & \cos\frac{gt}{\hbar} \end{bmatrix}$$

$$|\psi(t)\rangle = e^{-i\frac{\hat{H}}{\hbar}t}|\psi(0)\rangle = \begin{pmatrix} \cos\frac{gt}{\hbar} \\ -i\sin\frac{gt}{\hbar} \end{pmatrix} e^{i\frac{h}{\hbar}t}$$

$\boxed{5}$ a) The system is a harmonic oscillator with $\omega = \sqrt{\frac{k}{m}}$. In rescaled basis.

$$\psi_0(X) = \frac{1}{\pi^{1/4}} e^{-X^2/2}$$

Using $X = x\sqrt{\frac{m\omega}{\hbar}}$ and re-normalizing, we find

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$$

b) The original Hamiltonian is

$$\hat{H}_0 = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

Potential due to added Force: $V = -Fx$

New potential:

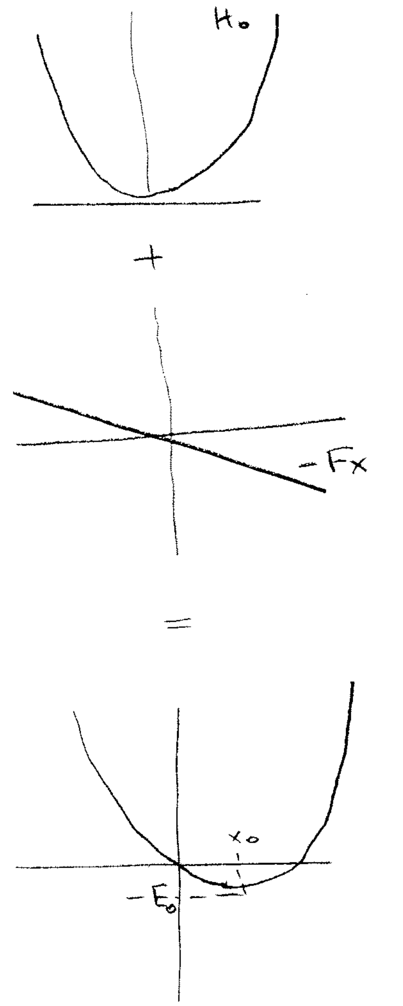
$x_0 = \frac{F}{k}$ (new equilibrium point)

E_0 (constant term, doesn't affect physics)

$$\hat{H} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} - Fx$$

$$= \frac{p^2}{2m} + \frac{m\omega^2\left(x - \frac{F}{m\omega^2}\right)^2}{2} - \frac{F^2}{2m\omega^2}$$

$$= \frac{p^2}{2m} + \frac{m\omega^2 (x-x_0)^2}{2} - E_0$$



This is the same Harmonic potential, but shifted by x_0.

$$\psi_0'(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega(x-x_0)^2}{2\hbar}}$$

c) $$pr = \left|\int \psi_0(x)\,\psi(x)\,dx\right|^2 =$$

$$= \frac{m\omega}{\pi\hbar}\left|\int e^{-\frac{m\omega}{2\hbar}[x^2+(x-x_0)^2]^2} dx\right|^2$$

$$= \frac{m\omega}{\pi\hbar}\left|\int e^{-\frac{m\omega}{2\hbar}[x^2 - x x_0 + \frac{x_0^2}{2}]^2} dx\right|^2$$

$$= \frac{m\omega}{\pi\hbar}\left|\int e^{-\frac{m\omega}{\hbar}\left[\left(x-\frac{x_0}{2}\right)^2 + \frac{x_0^2}{4}\right]} dx\right|^2$$

$$= \frac{m\omega}{\pi\hbar}\left|e^{-\frac{m\omega}{\hbar}\frac{x_0^2}{4}}\sqrt{\frac{\pi\hbar}{m\omega}}\right|^2 = e^{-\frac{m\omega}{\hbar}\frac{x_0^2}{2}} = e^{-\frac{F^2}{2\hbar m\omega^3}}$$

d) Let $\hat{y} = \hat{x} - x_0$. The new potential can then be written as

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{y}^2}{2} - E_0$$

From class we know:

$$\frac{d\langle y\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{y}]\rangle = \frac{\langle p\rangle}{m}$$

(because $[\hat{y},\hat{p}] = [\hat{x},\hat{p}] = i\hbar$)

$$\frac{d\langle p\rangle}{dt} = \frac{i}{\hbar}[\hat{H},\hat{p}] = -m\omega^2\langle y\rangle$$

Solve this system of differential equations for $\langle y \rangle$ and $\langle p \rangle$ with initial values $\langle y \rangle_0 = -x_0$, $\langle p \rangle_0 = 0$

$$\langle \ddot{y} \rangle = -\omega^2 \langle y \rangle$$

$$\Rightarrow \langle y \rangle = \langle y \rangle_0 \cos \omega t = -x_0 \cos \omega t$$

$$\langle p \rangle = m \langle \dot{y} \rangle = m x_0 \sin \omega t$$

Thus, $\langle x \rangle = \langle y \rangle + x_0 = x - x_0 \cos \omega t$

Evolution of expectation values reproduces classical!

$\boxed{6}$ $$\hat{H} = -\vec{\mu}\vec{B} = -\frac{ge}{2mc}\vec{B}\hat{\vec{S}} = -\frac{ge}{2mc}\frac{\hbar}{2}\vec{B}\hat{\vec{\sigma}},$$

where $\hat{\vec{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ are Pauli operators

Evolution under magnetic field

$$e^{-\frac{i}{\hbar}\hat{H}t} = e^{\frac{iget}{4mc}\vec{B}\vec{\sigma}}$$

Let $\frac{geBt}{4mc} = \Theta$

- If B is along x

$$e^{\frac{i}{\hbar}\hat{H}t} = e^{i\Theta\hat{\sigma}_x} = e^{i\Theta}\begin{pmatrix}1\\1\end{pmatrix}(1\ 1) + e^{-i\Theta}\begin{pmatrix}1\\-1\end{pmatrix}(1\ -1) = \begin{pmatrix}\cos\Theta & i\sin\Theta\\ i\sin\Theta & \cos\Theta\end{pmatrix}$$

$$e^{i\Theta\hat{\sigma}_x} = i\hat{\sigma}_x \quad \text{if} \quad \Theta = \pi/2$$

- If B is along y

$$e^{\frac{i}{\hbar}\hat{H}t} = e^{i\Theta\hat{\sigma}_y} = e^{i\Theta}\begin{pmatrix}1\\i\end{pmatrix}(1\ -i) + e^{-i\Theta}\begin{pmatrix}1\\-i\end{pmatrix}(1\ i) = \begin{pmatrix}\cos\Theta & +\sin\Theta\\ -\sin\Theta & \cos\Theta\end{pmatrix}$$

$$e^{i\Theta\hat{\sigma}_y} = i\hat{\sigma}_y \quad \text{if} \quad \Theta = \pi/2$$

- If B is along z

$$e^{\frac{i}{\hbar}\hat{H}t} = e^{i\Theta\hat{\sigma}_z} = \begin{pmatrix}e^{i\Theta} & \\ & e^{-i\Theta}\end{pmatrix}$$

$$e^{i\Theta\hat{\sigma}_z} = i\hat{\sigma}_z \quad \text{if} \quad \Theta = \pi/2$$

In all cases, $\Theta = \pi/2$ corresponds to $t = \frac{2\pi mc}{geB}$