PHYS 443

Final exam

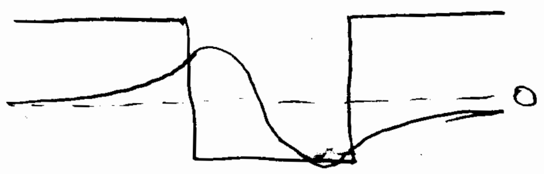
Solutions

1 $|30°\rangle = \begin{pmatrix} \cos 30° \\ \sin 30° \end{pmatrix}$

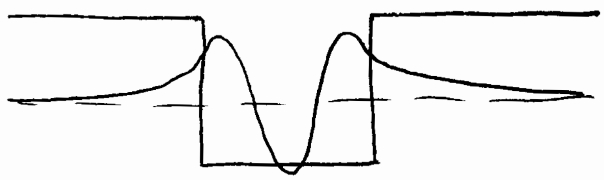
$pr(|R\rangle) = |\langle 30°|R\rangle|^2 = \left|\frac{1}{\sqrt{2}}(\cos 30° \;\; \sin 30°)\begin{pmatrix} 1 \\ i \end{pmatrix}\right|^2 = \frac{1}{2}(\cos^2 30° + \sin^2 30°) = \frac{1}{2}$

$pr(|L\rangle) = \frac{1}{2}$

2 Odd



Even



3 $L_x = \frac{L_+ + L_-}{2}$

$L_x |l=2, m=1\rangle = \frac{1}{2}\left(\sqrt{2\cdot 3 - 1\cdot 2}\;|l=2, m=2\rangle + \sqrt{2\cdot 3 - 1 \cdot 0}\;|l=2, m=0\rangle\right.$
$= |l=2, m=2\rangle + \frac{\sqrt{6}}{2}|l=2, m=0\rangle$

$L_x |l=2, m=2\rangle = \frac{1}{2}\left(0 + \sqrt{2\cdot 3 - 1\cdot 2}\;|l=2, m=1\rangle\right) = |l=2, m=1\rangle$

$\langle\psi|L_x|\psi\rangle = \frac{1}{2}\left(\langle 2,1| + \langle 2,2|\right)\left(|2,2\rangle + \frac{\sqrt{6}}{2}|2,0\rangle + |2,1\rangle\right) = 1.$

4 $|\alpha\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \cdots$

(explicit form of α's is unimportant)

$a^\dagger|\alpha\rangle = \alpha_0|1\rangle + \alpha_1\sqrt{2}\,|2\rangle + \cdots$

No $|0\rangle$ term $\Rightarrow$ $pr(|0\rangle) = 0.$

Or, simpler: $pr(|0\rangle) = |\langle 0|a^\dagger|\alpha\rangle|^2 = |\langle\alpha|\underbrace{a|0\rangle}_{=0}|^2 = 0$

5 Possible outcomes: $|\leftarrow\rangle = \frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}}$; $|\rightarrow\rangle = \frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}}$

$\langle\leftarrow|\Psi_{AB}\rangle = \frac{1}{3}\cdot\frac{1}{\sqrt{2}}\left(2|\uparrow\rangle - 2|\downarrow\rangle - |\downarrow\rangle\right) = \frac{1}{3\sqrt{2}}\left(2|\uparrow\rangle - 3|\downarrow\rangle\right)$

$pr(\leftarrow) = \frac{1}{3^2\cdot 2}(2^2 + 3^2) = \frac{13}{18}$

$\langle\rightarrow|\Psi_{AB}\rangle = \frac{1}{3}\cdot\frac{1}{\sqrt{2}}\left(2|\uparrow\rangle - 2|\downarrow\rangle + |\uparrow\rangle\right) = \frac{1}{3\sqrt{2}}\left(2|\uparrow\rangle - |\downarrow\rangle\right)$

$pr(\rightarrow) = \frac{1}{3^2\cdot 2}(2^2 + 1^2) = \frac{5}{18}$

$\boxed{6}$ In basis $\{|a_i\rangle\}$

$$\mathrm{Tr}\,(|u\rangle\langle v|) = \sum_i \langle a_i|u\rangle\langle v|a_i\rangle = \langle v|\left(\sum_i |a_i\rangle\langle a_i|\right)|u\rangle =$$
$$= \langle v|\hat{\mathbb{1}}|u\rangle = \langle v|u\rangle$$

$\boxed{7}$ a)

$$L_x \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \hbar \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} y = x\sqrt{2} \\ x + z = y\sqrt{2} \\ y = z\sqrt{2} \end{cases} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix} \begin{matrix} \leftarrow m=1 \\ \leftarrow m=0 \\ \leftarrow m=-1 \end{matrix}$$

norm $= 1$ ✓

b) $$\hat{H} = -B\hat{\mu}_z = -B\frac{e}{2m_e c}\hat{l}_z = -\hbar\Omega\sum_{m=-1}^{m} m\,|1,m\rangle\langle 1,m| \qquad \left(\Omega = \frac{Be}{2m_e c}\right)$$

$$|\Psi(t)\rangle = e^{-i\frac{H}{\hbar}t}|\Psi_0\rangle = \begin{pmatrix} \frac{1}{2}e^{i\Omega t} \\ 1/\sqrt{2} \\ \frac{1}{2}e^{-i\Omega t} \end{pmatrix}$$

c) For $\Omega t = \frac{\pi}{2}$, $$|\Psi(t)\rangle = \begin{pmatrix} i/2 \\ 1/\sqrt{2} \\ -i/2 \end{pmatrix}$$

$$L_y \begin{pmatrix} i/2 \\ 1/\sqrt{2} \\ i/2 \end{pmatrix} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} i/2 \\ 1/\sqrt{2} \\ -i/2 \end{pmatrix} = \hbar\begin{pmatrix} -i/2 \\ -1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} = -\hbar\,|\Psi(t)\rangle \quad ✓$$