

Solutions

1) a) $\hat{A} = 2|30^\circ\rangle\langle 30^\circ| + 3|120^\circ\rangle\langle 120^\circ|$

b) $|120^\circ\rangle = \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}$ $|30^\circ\rangle = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$

$$\hat{A} = 2 \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix} + 3 \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix} = \frac{2}{4} \begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} + \frac{3}{4} \begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 9 & -\sqrt{3} \\ -\sqrt{3} & 11 \end{pmatrix}$$

c) $|\langle H|30^\circ\rangle|^2 = \left| \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \right|^2 = 3/4$

$$|\langle H|120^\circ\rangle|^2 = \left| \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \right|^2 = 1/4$$

d) $\langle \hat{A} \rangle = 2 \times \frac{3}{4} + 3 \times \frac{1}{4} = \frac{9}{4}$

$$\langle A^2 \rangle = 4 \times \frac{3}{4} + 9 \times \frac{1}{4} = \frac{21}{4}$$

$$\langle \Delta A^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2 = \frac{3}{16}$$

2

$$a) (\hat{A}^\dagger)^T = \begin{pmatrix} -3 & +4i \\ -4i & 3 \end{pmatrix} = \hat{A}$$

b) Characteristic polynomial

$$(-a-3)(3-a) = 16$$

$$a^2 - 9 = 16 \Rightarrow a = \pm 5$$

$$a_1 = 5 \rightarrow \text{eigenvector } \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2i \end{pmatrix} = |a_1\rangle$$

$$a_2 = -5 \rightarrow \text{eigenvector } \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ i \end{pmatrix} = |a_2\rangle$$

$$c) \langle a_1 | a_1 \rangle = \frac{1}{5} (1 \ 2i) \begin{pmatrix} 1 \\ -2i \end{pmatrix} = 1$$

$$\langle a_1 | a_2 \rangle = \frac{1}{5} (1 \ 2i) \begin{pmatrix} 2 \\ i \end{pmatrix} = 0$$

$$\langle a_2 | a_2 \rangle = \frac{1}{5} (2 \ -i) \begin{pmatrix} 2 \\ i \end{pmatrix} = 1$$

$$d) a_1 \langle a_1 | + a_2 \langle a_2 | = 5 \frac{1}{5} \begin{pmatrix} 1 \\ -2i \end{pmatrix} (1 \ 2i) - 5 \frac{1}{5} \begin{pmatrix} 2 \\ i \end{pmatrix} (2 \ -i) =$$

$$= \begin{pmatrix} 1 & 2i \\ -2i & 4 \end{pmatrix} - \begin{pmatrix} 4 & -2i \\ 2i & 1 \end{pmatrix} = \begin{pmatrix} -3 & 4i \\ -4i & 3 \end{pmatrix} \checkmark$$

$$e) e^{-\frac{i}{\hbar} \hat{H} t} = \frac{1}{5} e^{-5i\omega t} \begin{pmatrix} 1 & 2i \\ -2i & 4 \end{pmatrix} + \frac{1}{5} e^{5i\omega t} \begin{pmatrix} 4 & -2i \\ 2i & 1 \end{pmatrix}$$

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar} \hat{H} t} |H\rangle = \frac{1}{5} e^{-5i\omega t} \begin{pmatrix} 1 \\ -2i \end{pmatrix} + \frac{1}{5} e^{5i\omega t} \begin{pmatrix} 4 \\ 2i \end{pmatrix}$$

$$\text{@ } \omega t_0 = \frac{\pi}{10} \quad |\psi(t_0)\rangle = \frac{1}{5} \begin{pmatrix} -i \\ -2 \end{pmatrix} + \frac{1}{5} \begin{pmatrix} 4i \\ -2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3i \\ -4 \end{pmatrix}$$

$$|\langle +45^\circ | \psi(t_0) \rangle|^2 = \frac{1}{2} \cdot \frac{1}{25} \left| (1 \ 1) \begin{pmatrix} 3i \\ -4 \end{pmatrix} \right|^2 = \frac{1}{2}$$