# Phys 443. Final exam

## Solutions

1. a) No: $\hat{R} \to \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ while $\hat{R}^{+} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ so $\hat{R} \neq \hat{R}^{+}$

b) $\hat{H} = \hbar\omega \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ Characteristic equation: $\det \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} = 0$

$$\lambda^3 - 3\lambda - 2 = 0$$

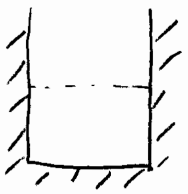
$$(\lambda + 1)^2 (\lambda - 2) = 0$$

Eigenvalue $\lambda = -1$ ($\times \hbar\omega$) : eigenstates $\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \to \frac{1}{\sqrt{2}} (|b\rangle - |c\rangle)$

$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \to \frac{1}{\sqrt{2}} (|a\rangle - |c\rangle)$

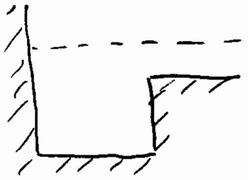
Eigenvalue $\lambda = 2$ ($\times \hbar\omega$) : eigenstate $\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \to \frac{1}{\sqrt{3}} (|a\rangle + |b\rangle + |c\rangle)$

3. a)

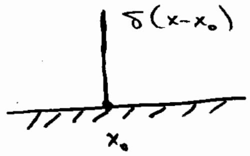


b) none (discontinuity)

c)



d)




In (d), the delta-function is positive because the second derivative of $\Psi(x)$ at $x_0$ is $+\infty$. The wave function is stationary if there are potential barriers somewhere to left and right.

2. Ex. 3.43: $\Psi_{t<0}(x) = \sqrt{k_0} \begin{cases} e^{-k_0 x} & x > 0 \\ e^{k_0 x} & x \leq 0 \end{cases}$ $\quad k_0 = \frac{m V_0}{\hbar^2}$

$$\Psi_{t>0}(x) = \sqrt{2k_0} \begin{cases} e^{-2k_0 x} \\ e^{2k_0 x} \end{cases}$$

(bound stationary solutions of the Schrödinger equation)

The probability that a bound state at $t=0$ will remain bound is

$$P = \left| \int \Psi_{t<0}(x)\, \Psi_{t>0}(x)\, dx \right|^2 = \left| 2\sqrt{2}\, k_0 \int_0^\infty e^{-3k_0 x}\, dx \right|^2 = \left| \frac{2\sqrt{2}}{3} \right|^2 = \frac{8}{9}$$

If the state does not remain bound, it will be localized at infinity at $t = \infty$. The probability to detect a finite position value is $P = 8/9$

4. $\hat{H} = J \vec{S}_A \vec{S}_B = J\left(\hat{S}_{x_A}\hat{S}_{x_B} + \hat{S}_{y_A}\hat{S}_{y_B} + \hat{S}_{z_A}\hat{S}_{z_B}\right)$

$$S_x \to \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_{x_A}S_{x_B} \to \frac{\hbar^2}{4} \begin{array}{c|cccc} & (\frac12\ \frac12) & (\frac12\ -\frac12) & (-\frac12\ \frac12) & (-\frac12\ -\frac12) \\ \hline (\frac12\ \frac12) & & & & 1 \\ (\frac12\ -\frac12) & & & 1 & \\ (-\frac12\ \frac12) & & 1 & & \\ (-\frac12\ -\frac12) & 1 & & & \end{array}$$

$$S_y \to \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_{y_A}S_{y_B} \to \frac{\hbar^2}{4}\begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}$$

$$S_z \to \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad S_{z_A}S_{z_B} \to \frac{\hbar^2}{4}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

$$\hat{H} \to \frac{J}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Eigenvalues $\left(\times \frac{J\hbar^2}{4}\right)$: $(1, 1, 1, -3)$

Eigenstates: $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$

$$e^{-i\hat{H}t/\hbar} = e^{-\frac{iJ\hbar t}{4}}\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} + e^{-i\frac{J\hbar t}{4}}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix} + e^{-\frac{iJ\hbar t}{4}}\begin{pmatrix}0&0&0&0\\0&1&1&0\\0&1&1&0\\0&0&0&0\end{pmatrix} + e^{+\frac{3iJ\hbar t}{4}}\begin{pmatrix}0&0&0&0\\0&1&-1&0\\0&-1&1&0\\0&0&0&0\end{pmatrix}$$

For $J\hbar t = \pi$:

$$e^{-i\hat{H}t/\hbar} = e^{-i\pi/4}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix} \equiv \hat{U}$$

If $|\psi\rangle = a|\uparrow\rangle + b|\uparrow\rangle$

$|\varphi\rangle = c|\uparrow\rangle + d|\downarrow\rangle$ (where $|\uparrow\rangle = |+\frac12\rangle$, $|\downarrow\rangle = |-\frac12\rangle$)

$$|\psi\rangle_A \otimes |\varphi\rangle_B = ac|\uparrow\uparrow\rangle + ad|\uparrow\downarrow\rangle + bc|\downarrow\uparrow\rangle + bd|\downarrow\downarrow\rangle$$

$$\hat{U}\left(|\psi\rangle_A \otimes |\varphi\rangle_B\right) = ac|\uparrow\uparrow\rangle + ad|\downarrow\uparrow\rangle + bc|\uparrow\downarrow\rangle + bd|\downarrow\downarrow\rangle =$$

$$= |\varphi\rangle_A \otimes |\psi\rangle_B$$