Kinetic Theory and Thermodynamics

Handout 4:

The thermal diffusion equation is

$$\boxed{\frac{\partial T}{\partial t} = \frac{\kappa}{C}\nabla^2 T}$$

where C is a heat capacity *per unit volume.*

The one-dimensional thermal diffusion equation is then

$$\frac{\partial T}{\partial t} = \frac{\kappa}{C}\frac{\partial^2 T}{\partial x^2}.$$

Spherical puddings!

One can also solve the (rather forbidding looking) problem of the thermal diffusion equation in a system with spherical symmetry. In spherical polars, we have in general

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}$$

so that if T is not a function of θ or ϕ we can write

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)$$

and hence the diffusion equation becomes

$$\boxed{\frac{\partial T}{\partial t} = \frac{\kappa}{C}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)}$$

In the lecture, we will show how we can transform this to a one-dimensional diffusion equation. This is accomplished using a substitution

$$T(r,t) = T_1 + \frac{\chi(r,t)}{r}$$

which reduces the problem to

$$\frac{\partial \chi}{\partial t} = \frac{\kappa}{C}\frac{\partial^2 \chi}{\partial x^2}.$$

We will solve this for the case of a spherical pudding of radius a at temperature T_0 which is placed into an oven at temperature T_1 at time $t = 0$. We will show that the temperature of the pudding ($r \le a$) behaves as

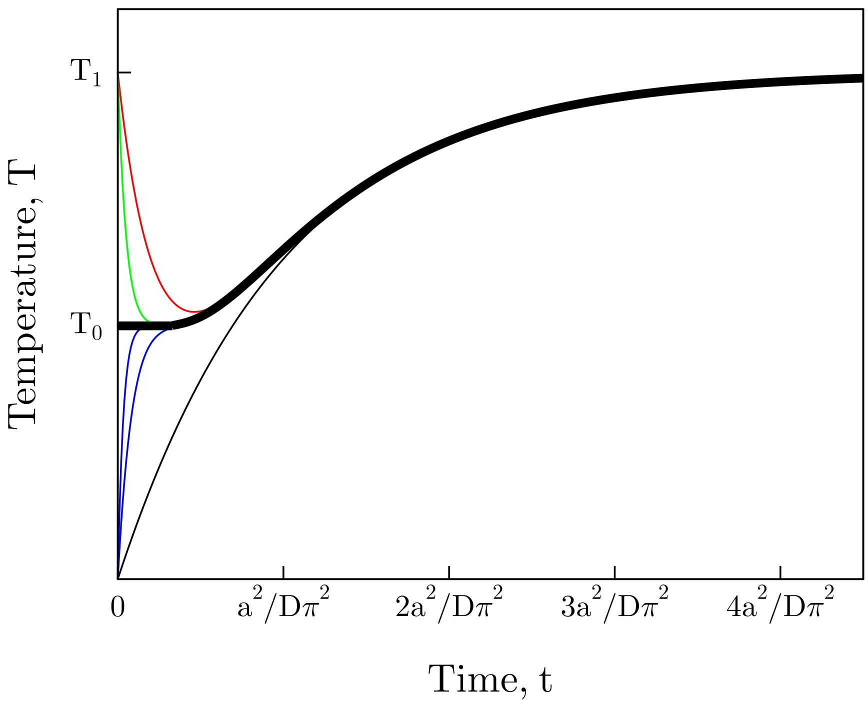
$$T(r,t) = T_1 + \frac{2a}{\pi}(T_1 - T_0)\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\sin(n\pi r/a)}{r}\mathrm{e}^{-D(n\pi/a)^2 t}.$$

where $D = \kappa/C$ and that the centre of the pudding has temperature

$$T(0,t) = T_1 + 2(T_1 - T_0)\sum_{n=1}^{\infty}(-1)^n \mathrm{e}^{-D(n\pi/a)^2 t}.$$

This becomes dominated by the first exponential in the sum as time t increases.

Exercise: Show that, in this problem, the temperature T gets 90% of the way from T_0 to T_1 after a time $\sim a^2 \ln 20/\pi^2 D$.



The sum of the first few terms of $T(0,t) = T_1 + 2(T_1 - T_0)\sum_{n=1}^{\infty}(-1)^n \mathrm{e}^{-D(n\pi/a)^2 t}$ are shown, together with $T(0,t)$ evaluated from all terms (thick solid line). The sums of only the first few terms fail near $t = 0$ and one needs more and more terms to give an accurate estimate of the temperatures as t gets closer to 0 (although this is the region where one knows what the temperature is anyway!)

Newton's law of cooling

This states that the heat loss of a solid or liquid surface (a hot central heating pipe or the exposed surface of a cup of tea) to the surrounding gas (usually air, which is free to convect the heat away) is proportional to the area of contact times the temperature difference between the solid/liquid and the gas.

Example: A polystyrene cup of tea at temperature T_{hot} at $t = 0$ stands for a while in a room with air temperature T_{air}. The temperature T of the cup of tea therefore falls behaves as $T = T_{\text{air}} + (T_{\text{hot}} - T_{\text{air}})\mathrm{e}^{-\lambda t}$ where λ depends on the surface area exposed to the air, the heat capacity of tea, the mass of tea in the cup and probably some correction factor for heat lost through the polystyrene cup.