Second year lectures on Electromagnetism.
Problem Set 5

Hilary term 2006.

1. A transient imbalance of charge will produce a transient electric field in an electrical conductor. Show that such charges and the associated electric field will decay exponentially with time with time constant $\tau = \epsilon_r\epsilon_0\rho$ where ρ is the resistivity.

   Deduce values for the relaxation time τ of a plastic dielectric with resistivity $10^{15}\Omega m$ and of copper with resistivity of $10^{-8}\Omega m$. Comment on the possible limitations of your calculation in the case of the good metal.

2. Write down Maxwell's equations for an isotropic conducting media containing no free charges. In the situation where you can neglect the displacement current, show that the current density $\mathbf{J}$ is given by

$$\nabla^2\mathbf{J} = \mu_r\mu_0\sigma\frac{\partial\mathbf{J}}{\partial t}$$

   where μ_r is the relative permeability and σ the conductivity.

   By considering solutions of the form

$$J_x = J_0 exp\left\{j\omega\left(t - \frac{nz}{c}\right)\right\}$$

   for the current of angular frequency ω flowing in the x direction near the plane surface ($z = 0$) of a semi-infinite conductor, where n is complex, show that the amplitude varies with depth as

$$exp\left(-\frac{z}{\delta}\right)$$

   where δ is the 'skin depth' and derive an expression for δ.

   By integrating in a direction perpendicular to the surface, show that the effective resistance per unit length of a circular conductor of radius $r (r \gg \delta)$ is

$$\frac{\rho}{2\pi r\delta}$$

   where ρ is the resistivity.

   How does the skin effect influence the design of aluminium overhead power cables operating at 50 Hz.

   [The electrical resistivity of aluminium is $1.5 \times 10^{-8}\Omega m$.]

3. The region between the plates of a parallel plate capacitor is evacuated and filled with a uniform space charge of free electrons of density N together with the same density of positive ions. An ac potential V=V$_0$cosωt is applied. Find an expression for the displacement of the electrons in the absence of scattering or damping. Show that the capacitor behaves as if it is parallel by an inductor. Find the value of this inductor.

   Deduce the impedance of the capacitor when $\omega = \omega_p$, the plasma frequency of the electrons where $\omega_p^2 = Ne^2/m\epsilon_0$.

   Explain why the positive ions may be ignored throughout.

4. Using your expression for the displacement of the electrons deduced in the previous problem, find an expression for the polarisation density as the electrons oscillate. Show thereby that the refractive index n experienced by an electromagnetic wave of frequency ω propagating in the plasma is given by

$$n = \sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}$$

Deduce the wavevector k for a wave of frequency ω. Show that when $\omega > \omega_p$, the product of the group and phase velocities equals c^2.

Discuss the form of k when $\omega < \omega_p$. What does this mean for the nature of the wave. Why does it attenuate? Is the energy being absorbed in the medium?

5. As an alternative approach to the electrodynamics of a plasma consider the response of a free electron to an oscillating electric field of the form $E = E_0 \exp(i\omega t)$. Show that (neglecting ionic currents) the conductivity of a low density neutral plasma with N free electrons per unit volume may be expressed as $\sigma = -iNe^2/m_e\omega$. Hence find again the frequency below which electromagnetic waves cannot propagate in the plasma.

   During a re-entry of the space-shuttle it was observed that radio communications at frequencies below 10 GHz were interrupted. Calculate the electron density in the neighbourhood of the shuttle.

6. A plane electromagnetic wave falls normally on the surface of a good conductor. Show that the energy lost per unit area may be approximated as $H^2/2\sigma\delta$ where δ is the skin depth and H is the magnitude of the magnetic field at the surface. Putting in numbers for a good metal estimate the fractional energy loss at such a metallic mirror at radio and at optical frequencies.

7. In the neighbourhood of the earth the solar flux is about 1400 watts m^{-2}. Calculate the maximum mass per unit area of a perfectly reflecting "sail" which is able to overcome solar gravity without the benefit of angular velocity about the sun. Compare this mass per unit area with that of a copper film whose thickness is sufficient to achieve the necessary reflectivity in the optical region. The density of copper is $9000 kg/m^3$.

8. (a) Explain the significance of the real and imaginary parts, n' and n'', of a complex refractive index.

   (b) A dilute gas is composed of N molecules per unit volume. The molecules may be considered as lightly damped harmonic oscillators of charge e and mass m with resonant angular frequency ω_0 and damping constant γ, where γ has the dimensions of angular frequency. Show that the complex refractive index n of the gas is given by

$$n^2 - 1 = \frac{Ne^2}{m\epsilon_0} \frac{1}{[(\omega_0^2 - \omega^2) + i\gamma\omega]}.$$

   What is the phase difference between an oscillator and the electric field acting on it

   i. for $\omega \ll \omega_0$

   ii. for $\omega \gg \omega_0$

   iii. for $\omega = \omega_0$?

   Assuming that

$$\frac{N_e^2}{m\epsilon_0\omega_0\gamma} \ll 1$$

   derive approximate expressions for n' and n''. Make a rough sketch showing how n' and n'' depend on ω in the vicinity of the resonance. Show that the full width at half maximum of the curve $k(\omega)$ is γ. What is the velocity of propagation of a wave of frequency ω_0 in the medium?