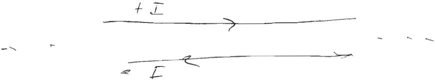
Simple transmission line



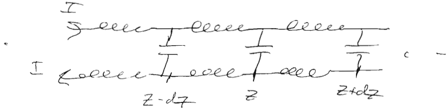
naive view is that this is a wire with zero resistance so the signal travels perfectly with no distortion!

But how fast?

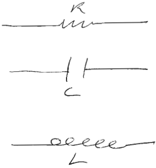
In circuit theory it is all instantaneous
and even with zero resistivity, as you learned last year & this.

Two pieces of metal in space have an inductance and a capacitance between them Even if $R = 0$

So actually our simple line has an inductance per unit length and a capacitance per unit length.



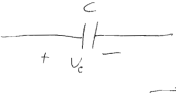
(1) Passive sign convention: Labelling scheme
Important to get transmission lines
understood



Simply stated: you define
voltage across two points
however you want.

Then current is defined to
enter the + terminal

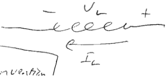
Reason
$I_C = C\frac{dV_C}{dt}$
with the
convention



current must
be labeled
flowing from left
to right

works backwards too: given current
voltage must be fixed.

Reason
$L\frac{dI_L}{dt} - V_L$
using passive sign convention



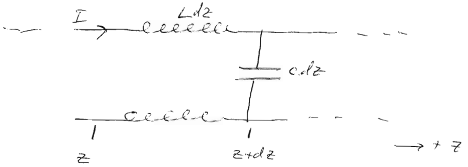
→ voltage must
be
labeled this way.

(2)

Take with you to Board!

Look carefully at a section of this

Line



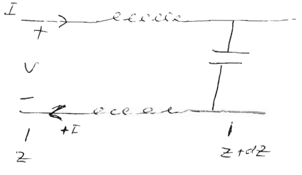
↑
start at location 'z'

and go in the positive 'z' direction

to z+dz

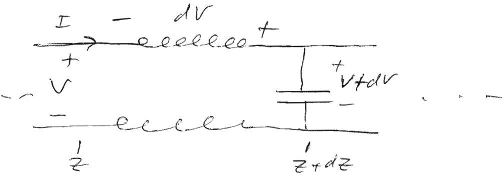
So the total inductance of our little segment is Ldz
and the capacitance is Cdz

Now define the voltage between the lines as 'V' at location z. Important!



When we head in the +z direction we need to define this as a positive voltage increase

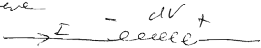
To match with our positive increase in z



so across the capacitor
we get V+dV

But Now notice !!
The labeling we are forced to obey because of our choice of +z does not obey passive sign convention for the inductor

we have



which is opposite.

So instead of $L\,dz\,\frac{dI}{dt} = dV$

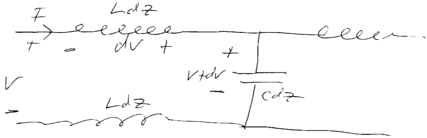
we must use ⇒

$$L\,dz\,dI/dt = -dV$$

So $-dV/dz = L\,dI/dt$ $L =$ ind/unit length

Next problem we have identified the current through the upper inductor

but not through the lower one

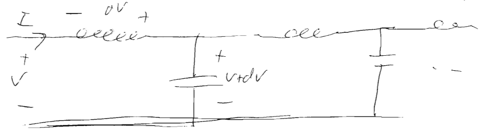


Pretty obvious from symmetry that we have to have $-I$ in the lower branch. and then the voltage drop is the same.

But this gives us the chance to

simplify our model by treating the lower branch as having zero inductance and the upper one as having all the inductance

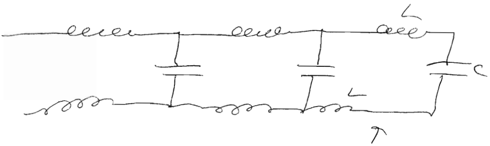
we will call it "L" anyways



like a line over an infinite ground plane.

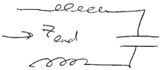
another proof →

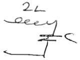
ASK If want to see Reason, if not skip this.

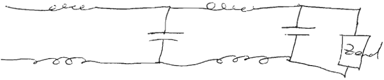


last part.

Combine impedances

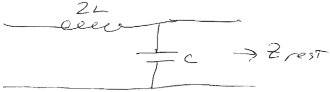


$Z_{end} = j\omega 2L + \frac{1}{j\omega C} \Rightarrow$ 



note that Z_{end} is in parallel with C

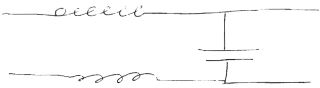
So this could always be modeled as



and since this is all general anyway I'm going to keep 'L' rather than carry around '2L' everywhere

Another Point

I've used the following model



but this model has 2-times the number of inductors as capacitors.

2-for-1 and this not really right

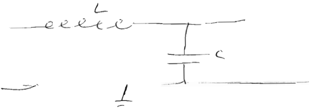
~~So really~~

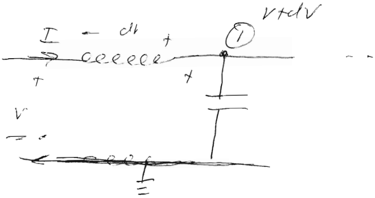
and I used 'L' the inductance per unit length on both inductors

But the per-unit inductance includes both when you actually calculate it.

So maybe I should have used 'L/2' instead of 'L'

Or "fixed" the model by doing what I am doing now:

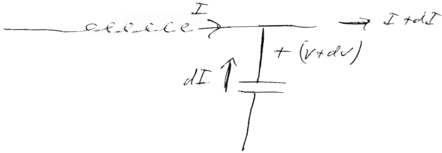




At node 1 There is a current which is through the capacitor that joins I at location $z+dz$

Once again, our choice of positive z means that we must add the incremental current to I at $z+dz$

add to I to keep positive



But notice that, accross the capacitor, we once again have violated passive sign convention

So $$-dI = C\,dz \frac{d}{dt}(V+dV)$$

Think of $dV \simeq \Delta V$
and $\frac{d}{dt}$ as $\frac{\Delta}{\Delta t}$

So $$\frac{d}{dt}(V + dV) \simeq \frac{\Delta V}{\Delta t} + \frac{\Delta V^2}{\Delta t}$$

but ΔV^2 is too small to worry about so:

$$-dI/dz = C\, dV/dt \quad (A)$$

recall that

$$-dV/dz = L\, dI/dt \quad (B)$$

(B)

$$-\frac{d^2V}{dz^2} = L \frac{d}{dz}(dI/dt)$$

$$-\frac{d^2V}{dz^2} = L \frac{d}{dt}(dI/dz)$$

now use (A)

$$-\frac{d^2V}{dz^2} = -LC \frac{d^2V}{dt^2}$$

So

$$\frac{d^2V}{dz^2} = LC\, d^2V/dt^2$$

wave equation with $v^2 = \frac{1}{LC}$

coaxial cable $C = \frac{2\pi\varepsilon}{\ln(b/a)}$ $L = \frac{\mu \ln(b/a)}{2\pi}$ so $v^2 = c^2$

From 1st year waves + normal modes

General Solution is

(1) $V(z,t) = F(z - vt) + G(z + vt)$

let $u = z - vt$
$w = z + vt$

so $\frac{du}{dt} = -v$ $\frac{du}{dz} = 1 = dw/dz$

$\frac{dw}{dt} = v$

using (1) let's take the time derivative

$$\frac{dV}{dt} = \frac{du}{dt}\frac{dF}{du} + \frac{dw}{dt}\frac{dG}{dw}$$

$$= -v\frac{dF}{du} + v\frac{dG}{dw}$$

But previously we concluded

$$\frac{dV}{dt} = -\frac{1}{C}\,dI/dz$$

so $-\frac{1}{C}\,dI/dz = -v\frac{dF}{du} + v\frac{dG}{dw}$

$$-\frac{1}{Cv}\,dI/dz = \frac{dF}{du} - \frac{dG}{dw} \qquad v = \frac{1}{\sqrt{LC}}$$

$$\sqrt{\frac{L}{C}}\,\frac{dI}{dz} = \frac{dz}{dz}\frac{dF}{du} - \frac{dz}{dz}\frac{dG}{dw}$$

multiply by one

$$\sqrt{\frac{L}{C}}\ \frac{dI}{dz} = \frac{dz}{du}\frac{dF}{dz} - \frac{dz}{dw}\frac{dG}{dz}$$

but $\frac{dz}{dw} = 1 = \frac{dz}{du}$

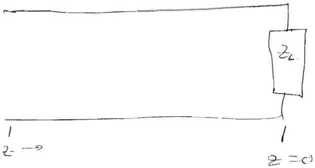
$$\sqrt{\frac{L}{C}}\ \frac{dI}{dz} = \frac{dF}{dz} - \frac{dG}{dz}$$

So we see that

$$Z_0 I = F(z - vt) - G(z + vt)$$

with $Z_0 = \sqrt{\frac{L}{C}}$

~~So Now a problem~~ Now a more practical Problem



Transmission Line with a given Load

what happens when such a voltage or current wave encounters a terminating impedance?

need a solution we can work with

complex exponential !!

$$V(z,t) = A e^{(j\omega t - jkz)} + A' e^{(j\omega t + jkz)}$$

of course $j - i = \sqrt{-1}$

which is fine as long as we have $\frac{\omega}{k} = v \qquad v^2 = \frac{1}{LC}$

This then does have the form of our General Solution $[F(z-vt) + G(z+vt)]$ and from previously we must have

$$Z_0 I(z,t) = A e^{(j\omega t - jkz)} - A' e^{(j\omega t + jkz)}$$

At $z = 0$ we have impedence Z_L and the way we defined our + current and voltage sense means passive sign convention holds

so $V = Z_L I$ (I + must)

so $$\frac{V(0,t)}{Z_0 I(0,t)} = \frac{Z_L}{Z_0} = \frac{A e^{j\omega t} + A' e^{j\omega t}}{A e^{j\omega t} - A' e^{j\omega t}}$$

and all the dependence drops out!

$$\frac{V}{Z_0 I} = \frac{Z_L}{Z_0} = \frac{A + A'}{A - A'}$$

we choose A so solve for ratio of

$$\frac{A'}{A} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$A \rightarrow$ incoming wave amplitude.

$A' \rightarrow$ returning or "reflected" wave

If $Z_L = Z_0$

$$\frac{A'}{A} = 0$$

No returning wave!

$Z_0 = \sqrt{\frac{L}{C}}$ is real

and Z_L can be complex
But to get no return wave
$Z_L \Rightarrow$ pure Resistance

If $Z_L = \infty$

$$\frac{A'}{A} = 1$$

reflected voltage is in same direction
reflection complete

$Z_L = 0$

$$\frac{A'}{A} = -1$$

again reflection is complete
But reflected wave is of opposite sense.

Z_0 is like having a pure load resistance at every point down the line, when you get to the end, if there is no match, this has to be communicated to the source.

Power?

$$P = VI$$

$$= \frac{\left(Ae^{j\omega t}e^{-jkz} + A'e^{j\omega t}e^{jkz}\right)\left(Ae^{j\omega t}e^{-jkz} - A'e^{j\omega t}e^{jkz}\right)}{Z_0}$$

$$= \frac{1}{Z_0}\left[A^2 e^{j2\omega t}e^{-j2kz} - AA'e^{j2\omega t} + A'Ae^{j2\omega t} - A'^2 e^{j2\omega t}e^{jkz}\right]$$

$$= \frac{1}{Z_0}\left[A^2 \exp(j2\omega t - j2kz) - A'^2 \exp(j2\omega t + j2kz)\right]$$

we see power also has components

traveling forward & backward

so when $A' = 0$ there is no power returning down the line.

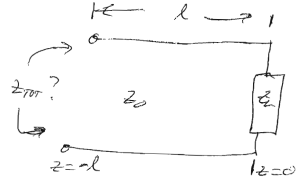
Careful! I used complex voltage.

So really, to be rigorous, would have needed to take the real part of V & the real part of I

but this illustrates the point nonetheless.

Though the Average Power trick still works $P_{ave} = \frac{1}{2}(VI^*)$

what is impedance of a stub?



$$V = A\exp(i\omega t + ikl) + A'\exp(i\omega t - ikl)$$

$$Z_0 I = A\exp(i\omega t + ikl) - A'\exp(i\omega t - ikl)$$

$$Z_{TOT} = V/I = Z_0 \frac{A e^{ikl} + A' e^{-ikl}}{A e^{ikl} - A' e^{-ikl}}$$

which is a bit of a mess in general.

but what if $lk = \frac{2\pi}{\lambda} l$

$= \pi/2$

in other words, what if $l = \lambda/4$?

then $e^{ikl} = e^{i\pi/2} = i$

$e^{-ikl} = e^{-i\pi/2} = -i$

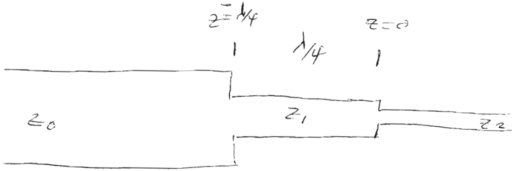
$$Z_{TOT} = Z_0 \frac{i(A - A')}{i(A + A')} = \cancel{\frac{1}{Z_0}} \qquad \frac{(A+A')}{A-A'} = \frac{Z_L}{Z_0}$$

$$= Z_0\left(\frac{Z_0}{Z_L}\right)$$

$$= Z_0^2/Z_L$$

(8)

So what?



What does the point at $z = -\frac{\lambda}{4}$ look like?

well Z_1 looks like it is terminated with an impedance of Z_2

so

at $z = -\lambda/4$

$$Z_{rat} = \frac{Z_1^2}{Z_2}$$

assume Z_2 is fixed, & that we can

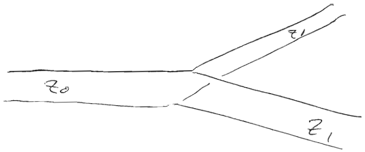
choose Z_1, if

$$Z_0 = Z_1^2/Z_2$$

matched impedance 4 the lines!

$$Z_0 Z_2 = Z_1^2$$

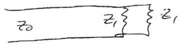
$$Z_1 = \sqrt{Z_0 Z_2} \Rightarrow$$ no reflections down Z_0!!



Sometimes, easier than you think.

What would Z_1 have to be so there are no reflections back down Z_0?

Answer $Z_1 = 2Z_0$ acts like

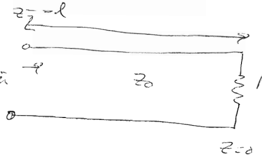


Two terminating Z_1's in parallel

Note: I was careful to always use Z_0, Z_L ...

remember, impedances can be complex or imaginary. Every formula still works!

General case with real termination



$$Z_{in} = \frac{V}{I} = Z_0 \frac{A e^{ikl} + A' e^{-ikl}}{A e^{ikl} - A' e^{-ikl}}$$

But if $Z_L = R$

$$\frac{A'}{A} = \frac{R - Z_0}{R + Z_0}$$

$$Z_{in} = \frac{Z_0 \left(e^{ikl} + \frac{R - Z_0}{R + Z_0} e^{-ikl} \right)}{e^{ikl} + \frac{Z_0 - R}{R + Z_0} e^{-ikl}}$$

$$= \frac{Z_0 \left[(R + Z_0) e^{ikl} + (R - Z_0) e^{-ikl} \right]}{(R + Z_0) e^{ikl} + (Z_0 - R) e^{-ikl}}$$

$$Z_{in} = \frac{Z_0 \left[2R \cos kl + 2i Z_0 \sin kl \right]}{2i R \sin kl + 2 Z_0 \cos kl}$$

$$Z_{in} = Z_0 \frac{R \cos kl + i Z_0 \sin kl}{Z_0 \cos kl + i R \sin kl} = Z_0 \frac{R + i Z_0 \tan kl}{Z_0 + i R \tan kl}$$

$$Z_{in} = R \left(\frac{1 + j \frac{Z_0}{R} \tan kl}{1 + j \frac{R}{Z_0} \tan kl} \right)$$

Notice that

this is only real

when $\tan kl = 0, \pi/2, \pi, 3\pi/2$

$= n\pi/2 \quad n = 0, 1, 2,$

at $kl = \frac{2\pi l}{\lambda} = 0, \pi, 2\pi, 3\pi$

$= m\pi \quad m = 0, 1, 2, 3$

$\tan kl = 0$ and $Z_{in} = R$

at $kl = \pi/2, 3\pi/2, 5\pi/2$

$= (2m+1) \quad m = 0, 1, 2, 3$

$\tan kl \to \infty$

and $Z_{in} = R\left(\frac{Z_0/R}{R/Z_0}\right) = \frac{Z_0^2}{R}$

as found previously

at every other point we get a complex impedance which is

either ~~reactive~~ capacitive

or inductive

So this is how you can make almost any kind of passive circuit element you want in a transmission line.

When does this apply?
(meaning all this theory)
Easy Rule of Thumb

if the frequency of ω
is such that $\lambda \gtrsim l/10$

where l is the length of your
wires/traces on circuit board
you may need to consider impedance
matching in your system.

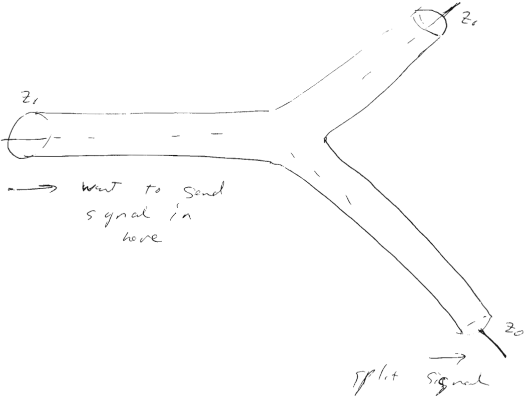
For a mobile phone;
$\sim 5cm$
So if $\lambda \approx 0.5 cm$
$\simeq 5 mm$

assuming the speed of light
$f \simeq 60 GHz$

Since mobile phone $\sim 3 GHz$ at
most they get away without this

Probably helps keep them cheap!

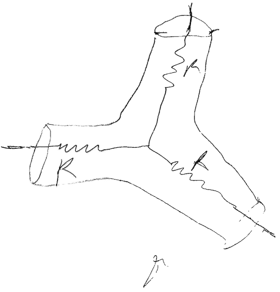
Final example If Time



But we do not want any back reflections

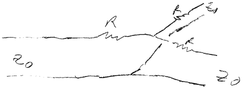
Why?

Because we do not want Sky 2 to know we have multiple sets connected.



connector

~~co~~ symmetric
That way can hook it up any
way we want



$$\text{Total} = R + \frac{(R+Z_0)(R+Z_0)}{2R+2Z_0}$$

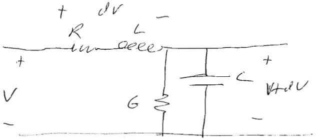
want total to equal Z_0

$$Z_0 = R + \frac{(R+Z_0)^2}{2(R+Z_0)}$$ → continue on board

$R = Z_0/3$ answer

Light bit for completeness
(off syllabus)

Electric wires have resistance
Dielectrics have losses



R - resistance/length

G = conductance/length.

Before we had

$$-\frac{dV}{dz} = L\,\frac{dI}{dt} \qquad -\frac{dI}{dz} = C\,\frac{dV}{dt}$$

Now we have

$$-\frac{dV}{dz} = RI + L\,\frac{dI}{dt}$$

$$-\frac{dI}{dz} = GV + C\,\frac{dV}{dt}$$

Using same procedure as before we get after algebra & calculus

$$\frac{d^2V}{dz^2} = RGV + (RC+LG)\frac{dV}{dt} + LC\frac{d^2V}{dt^2}$$

This is a damped travelling wave

of form

$$e^{-\alpha z} e^{i(\omega t - kz)}$$