Electric Potential Boundary Problems

Part I – Fields in the Presence of conductors.

Poisson and Laplace Equations

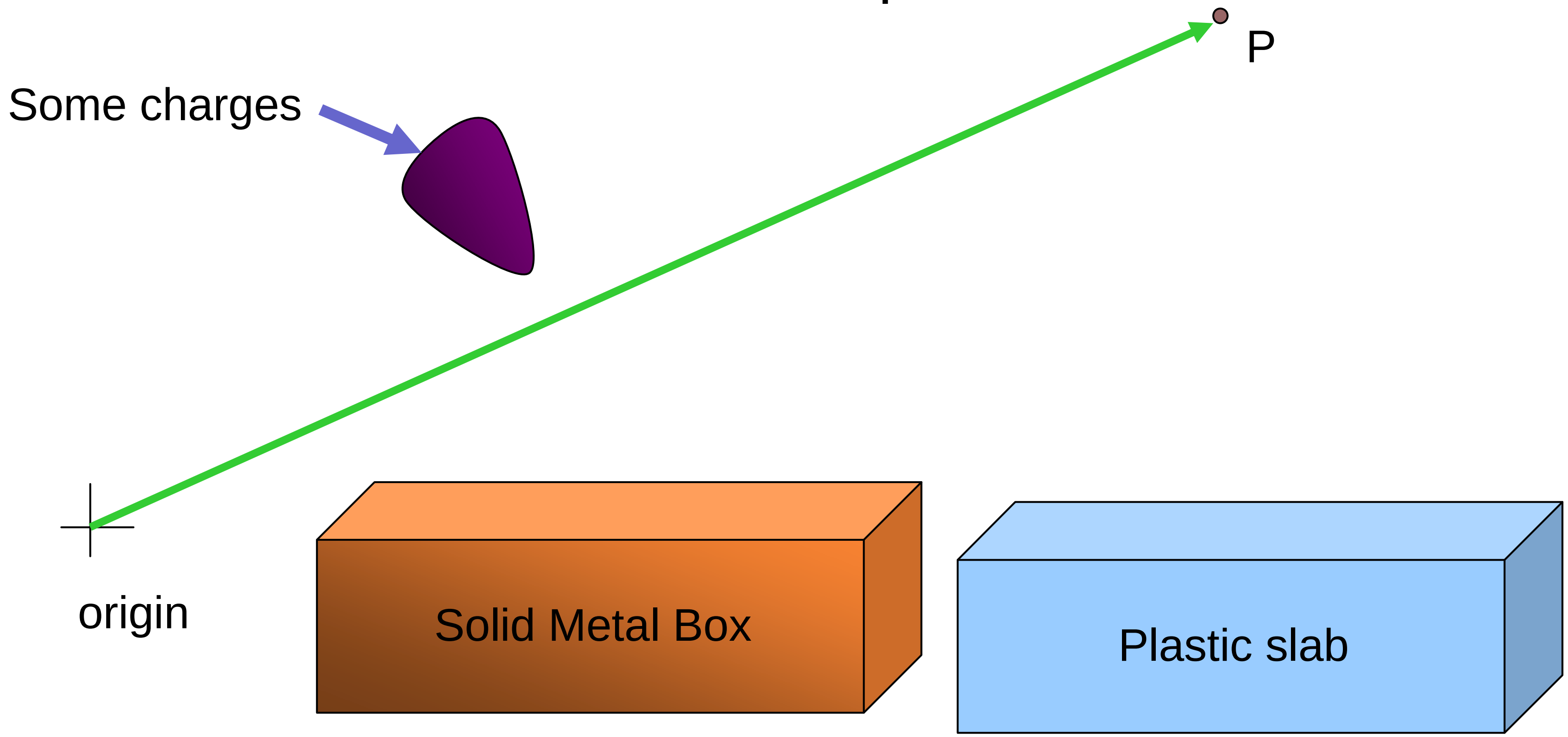
$$\nabla^2 V = -\rho(\vec{r}) \Big/ \varepsilon_0$$

Focus on the Laplace equation in an attempt to remain ON-SYLLABUS

$$\nabla^2 V = 0$$

Electrostatic Solutions in Space

The Problem:
What is the Electric Field at point ‘P’ ?



Electrostatic Solutions in Space

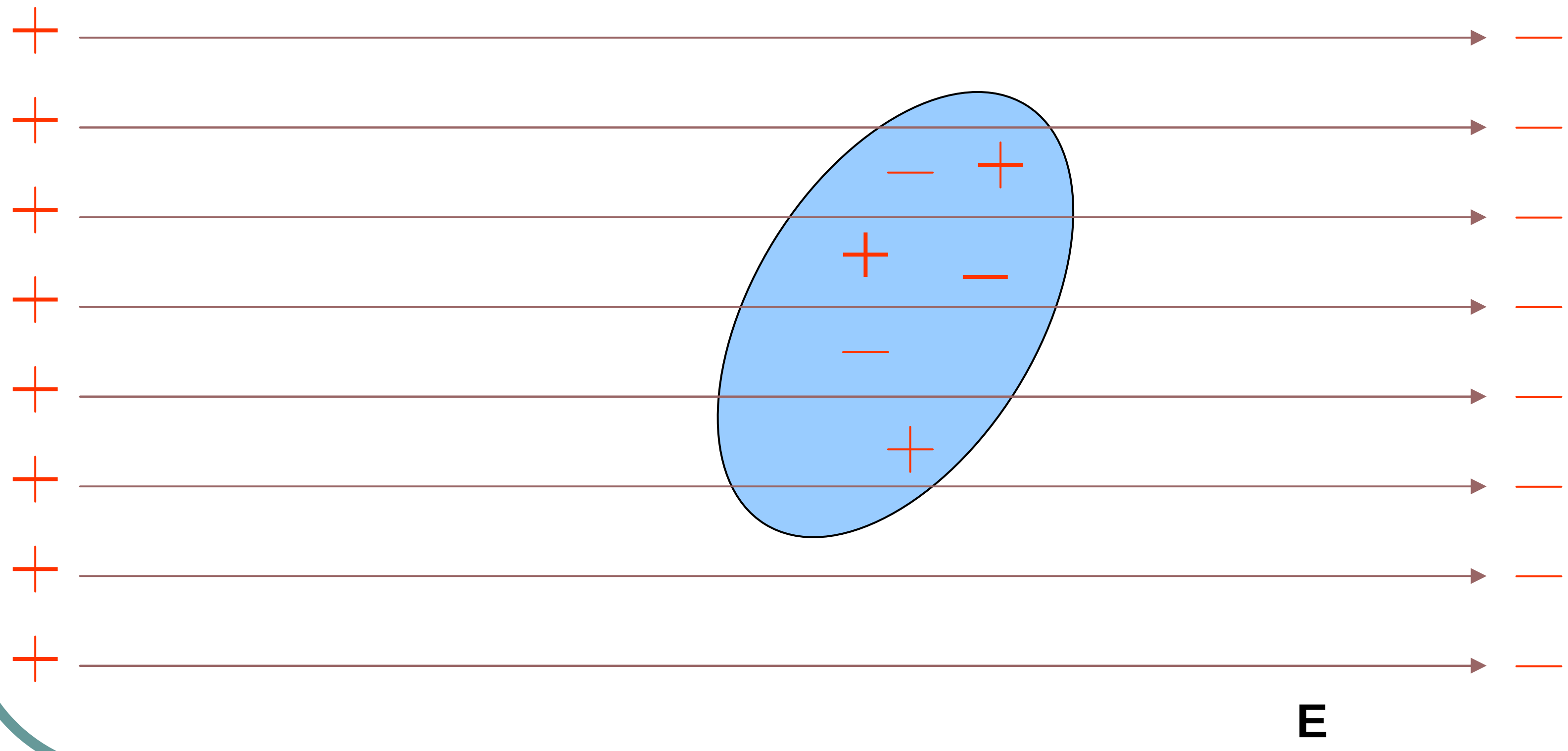
- For the moment, we will ignore the plastics
  - These are ‘dielectrics’ and we will be covering their properties next term.
  - Only worry about charge distributions and metal conductors
- Even better, we will start with metal conductors only.

Review of a Conductor's Electrostatic Properties

- Assumed to have very high conductivity.
- Charges freely move inside a conductor.
  - Means that all of the conductor must be at a single potential
  - If a charge is placed on a conductor it will distribute itself around the surface of the conductor.
- Electric Fields will not be present inside such a conductor at all.

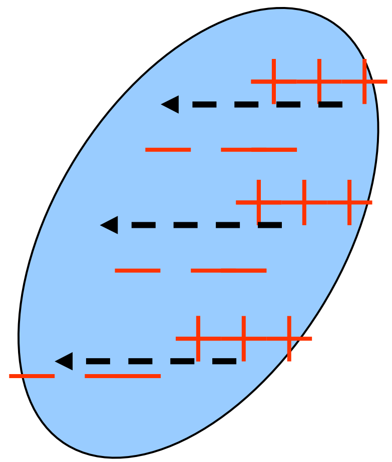
Conductor's properties

Start with an insulator with some charges embedded in it. Of course, those charges cannot move in an applied field.



Now allow charges to move

In the electric field, the charges will move to the edges of the conductor. But this creates a counter-field inside since the charges cannot leave the surface of the conductor.



E – Field from internal chares

End result

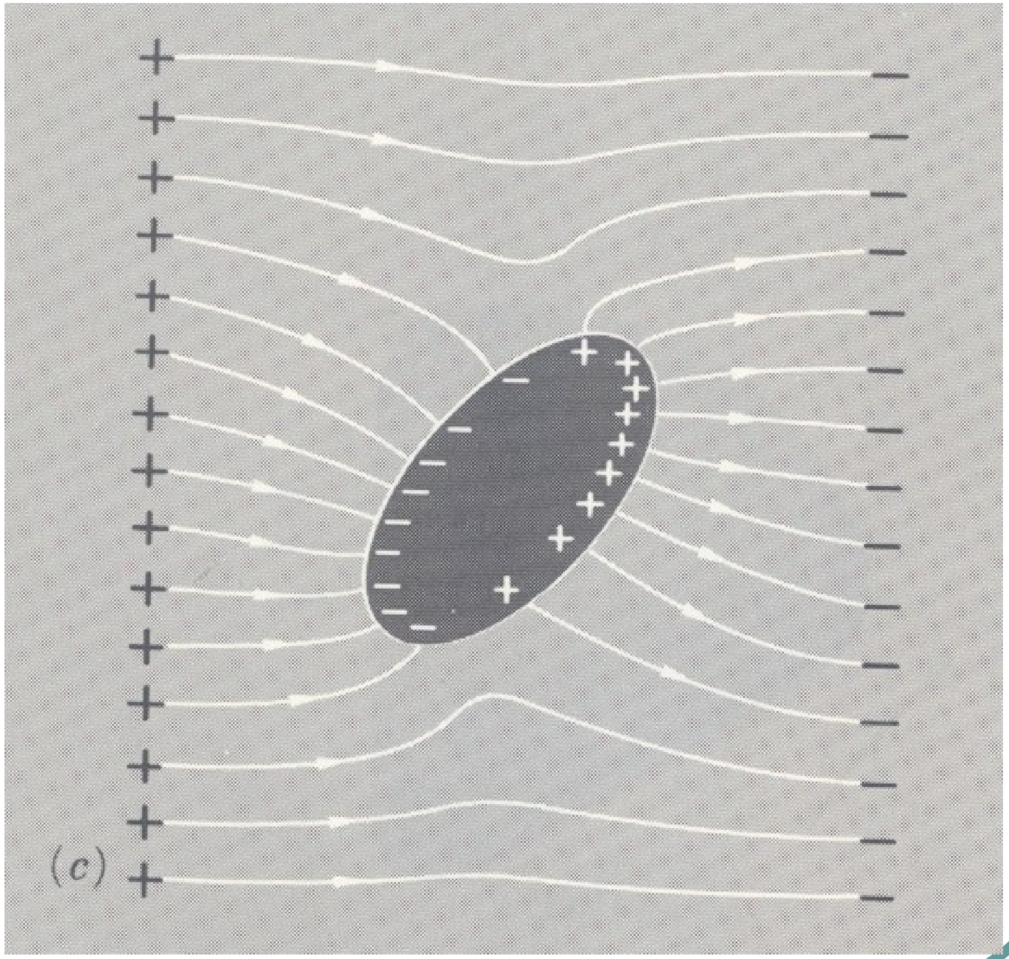
Final electric field is very different from where we started

It is no longer uniform.

But we know how the E-field behaves in the 'space between' conductors and charges.

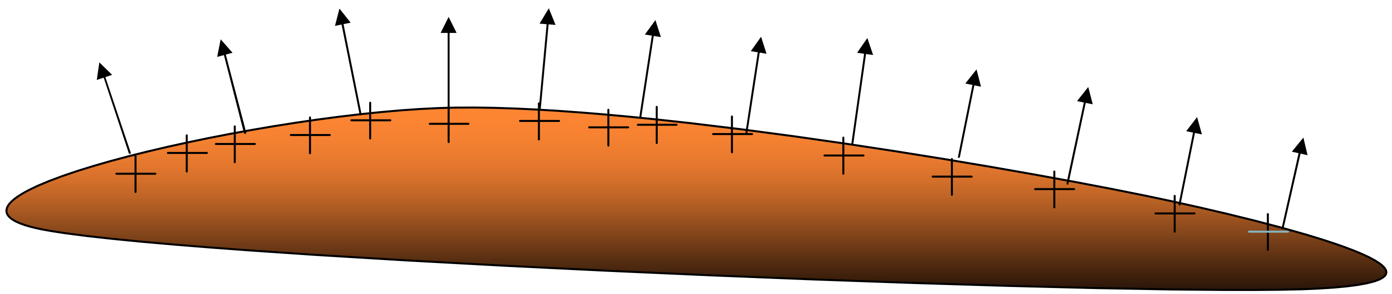
So how do we solve these

Boundary Value Problems!



Boundary Conditions: Conductor

- E-field always perpendicular to surface of conductor.
- E-field is zero inside the conductor.
- $\Delta \mathbf{E}_{perp} = \sigma_s$
  - The change in E-field is equal to the charge on the conductor's surface.



Laplace Equation:

Board Work

- Some obvious questions
  - What does it mean?
  - How can we use it?
- Lets start easy.
  - Cartesian coordinates
  - One-dimension

$$\nabla^2 V = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{\partial^2 V}{\partial x^2} = 0$$

1-Dim. Laplace equation

- In one dimension it is indeed trivial.
- But two items are worth thinking about.
  - 1). $V(x)$ is the average of $V(x+a)$ and $V(x-a)$ for any '*a*'.
  - 2). Laplaces equation tolerates *no maxima or minima*!
    - Extreme values of V must be at the end points.

Laplace Equation:

Board Work

$$\nabla^2 V = 0$$

- Let's start to get serious now.
  - Cartesian coordinates
    - But not too serious
  - Two dimensions

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Laplace Equation: 2-D

- So we see there is a similar rule
  - The potential at a point is equal to the average of the values at all points equidistant from our point in question.
- This rule also applies in 3-D
- **There are no Maxima or Minima away from the boundaries!**
  - Except for trivial solutions: zero or a constant.

Uniqueness: Laplace equation

- The solution to Lapace’s equation in a given volume is uniquely determined if either:
  - The Electric Potential, V, is specified on every boundary of that volume.

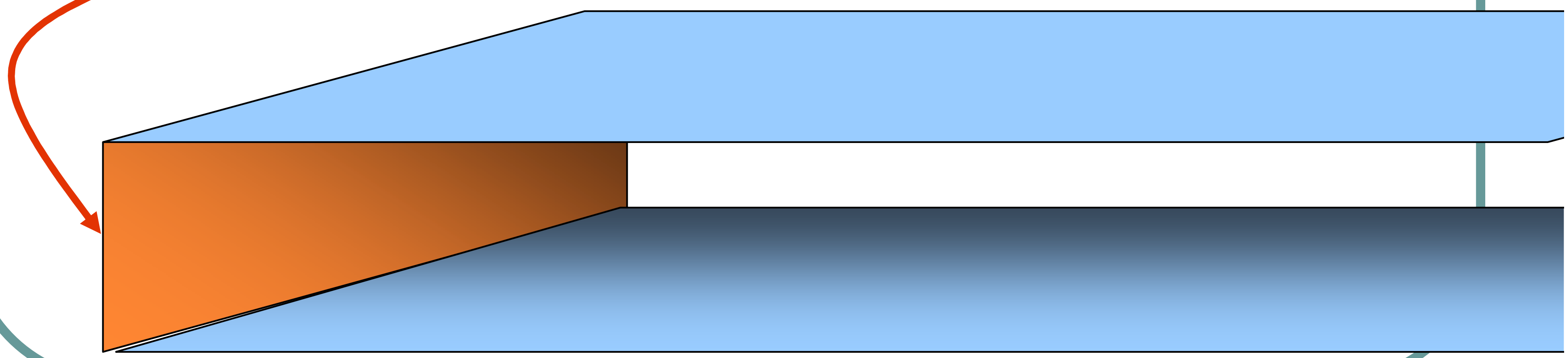
    Or
  - The total charge on each conductor bounding the volume is given.

Uniqueness: Laplace equation

- Another way to put this:
- It really doesn't matter what technique you use. Any function that:
  - Satisfies the Laplace equation, and
  - Meets *all* the boundary conditions
- **IS THE ONLY POSSIBLE SOLUTION!!**

Voltage Problem Example

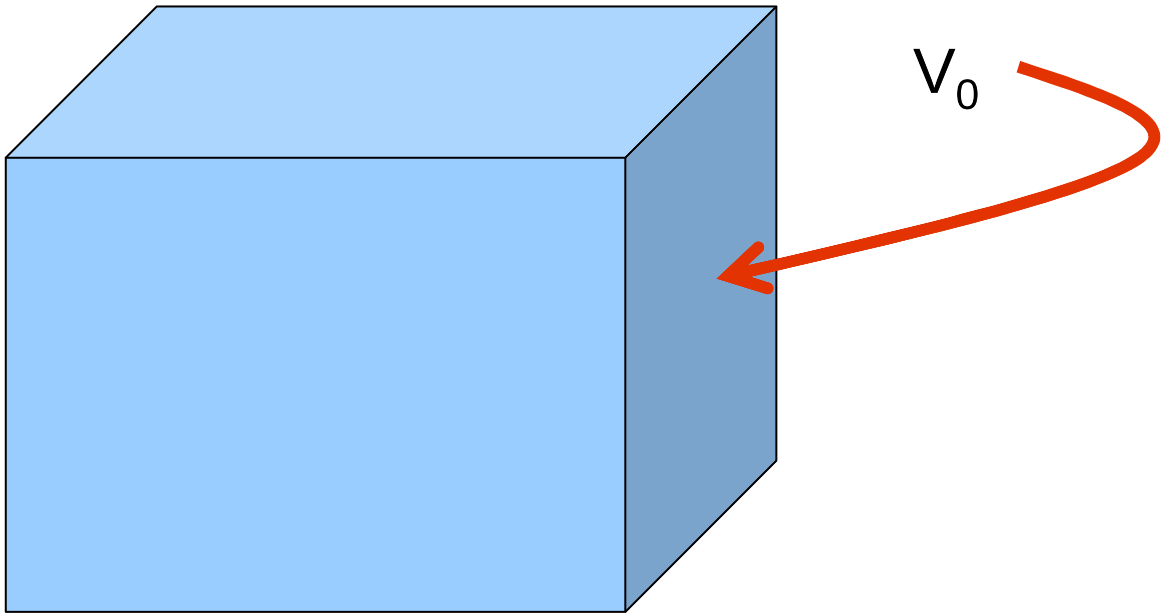
- Find the Electric Potential everywhere within the following configuration:
  - Two parallel metal semi-infinite plates held at V=0 volts and separated by a distance ‘a’ in meters.
  - A Third plate caps them off at the end and is held at V_0 volts.



Let the Buyer Beware!

- This technique would not have worked if we had grounded the cap and put a potential V_0 on one of the plates.
  - One cannot represent function that is unbounded at infinity as a sum of *discrete frequency* sines and cosines.
    - You can with a continuous set of frequencies → called a **Fourier Integral** (*not* Fourier series).
  - Series solutions always have bounding limitations.

Caveat Emptor



Solving for the interior of the box will work fine.

Solving for the outside of the box will not.