Electromagnetism
Year 2

Lecture 1
Properties of Electric Fields

Some Electrostatics Revision

- For the moment, Electric Charges are assumed to be stationary in the lab frame of reference
- Moving charges are a much more difficult problem…as you will eventually discover.
- The discussion proceeds from the experimental measurement of forces.

You must first do work to charge something up electrically.

One Animal was most definitely harmed in the making of these images.

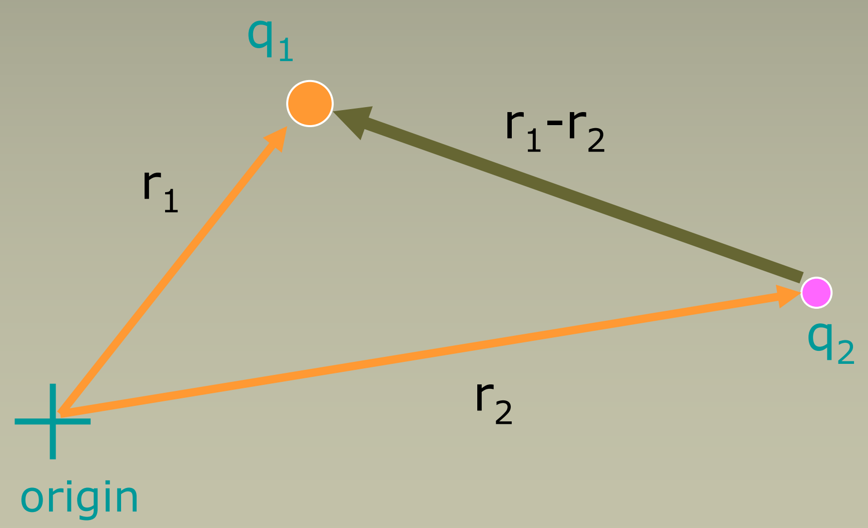
Some Basics on Electrostatics

- Ancient Experimental Observations:
  - There are two kinds of electric charge
  - When you have two massive objects that have this property, there is a measurable force between them
  - This force can be either attractive or repellent.
  - If the objects are sufficiently small compared to their distance. The force obeys an inverse square law. (Kind of like Gravity it seems.)
  - We all know and love this law!

Some Basics on Electrostatics

- The Force law:
- Conveniently, the sign of the force works correctly if we label our two 'kinds' of charge '+' and '-'

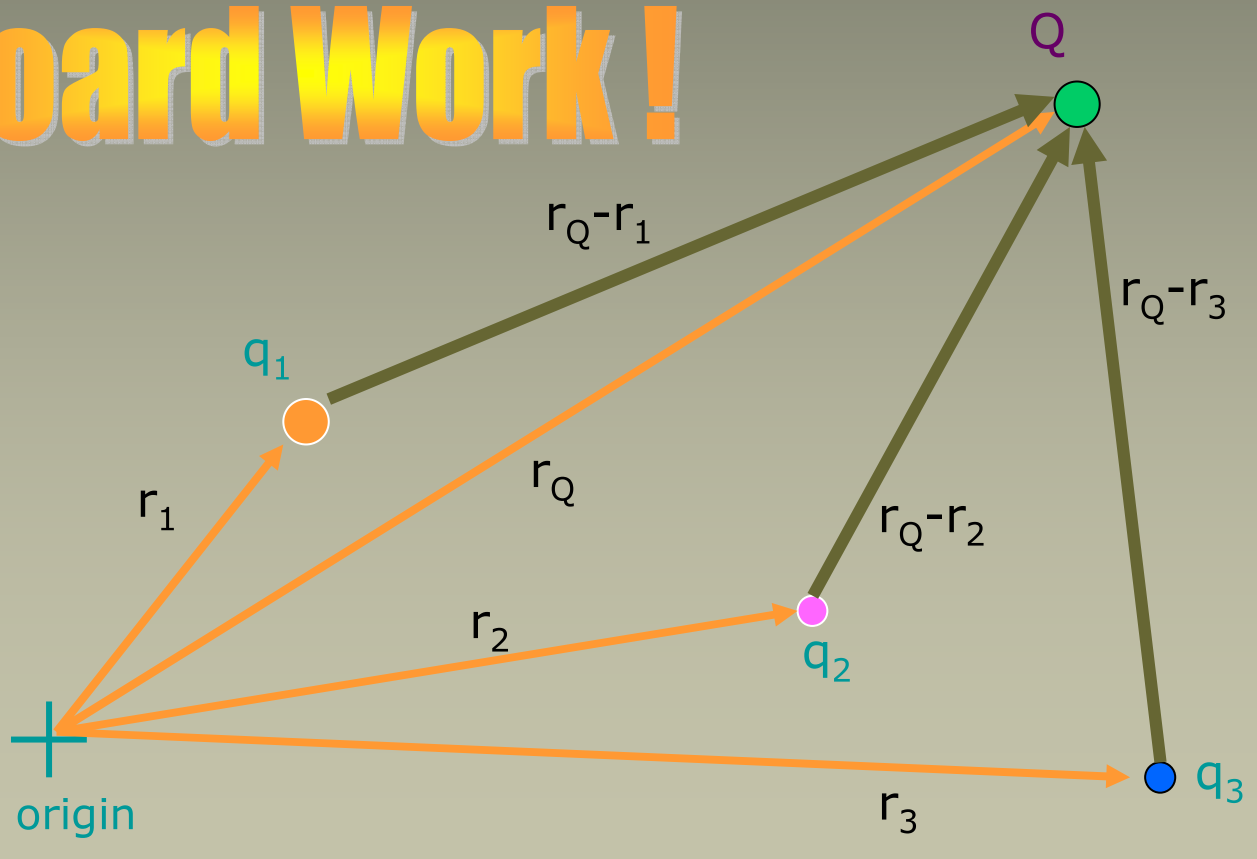
$$\vec{F} = \frac{q_1 q_2}{4\pi\varepsilon_0} \cdot \frac{(\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3}$$



Some Basics on Electrostatics

- What is the force on Q due to several other charges, $q_1 \ldots q_N$?

Board Work !



… and so on.

Force Linear wrt the Charge

All linear relations have this property.

Pull the common charge out of this large sum and we are left with the

"Electric Field"

caused by the sum of all charges sources which are causing a force to be evident on my test object.

$$\vec{E} = \sum_{all\ q's} \frac{q_1}{4\pi\varepsilon_0} \cdot \frac{(\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3}$$

E-field from Continuous charge

Derive the Integral form on the board!

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int_{all\,space} \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} \rho(x', y', z')\, dx' dy' dz'$$

Note that the integral is over the primed coordinates.

Now I shall show how complicated it can be with a trivial example.

The electric field due to a finite line of charge.

(show next slide)

Some Helpful Rules

- Find E-field at arbitrary point 'P' in the positive quadrant (or octant) and start with Cartesian coordinates.
- Use symmetry to simplify if you can!
- Draw r to point P and r' to an arbitrary place in the charged body.
  - Use the positive octant here too if you can!

Board Work !

Divergence and Curl of the E-field

- We have an expression for E.
- We can 'just' take it's divergence…

$$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial y}$$

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\varepsilon_0} \nabla \cdot \int_{all\,space} \frac{\vec{r} - \vec{r}'}{\left|\vec{r} - \vec{r}'\right|^3} \rho(\vec{r}') dV'$$

Notice that the vector derivative does **not** depend on the primed coordinates.

This is important for several next parts.

Board Work

Divergence of the E-field

$$\nabla \cdot \vec{E} = \frac{\rho(\vec{r})}{\varepsilon_0}$$

Differential form of Gauss's Law.

Integrate this equation over all space,

apply the Divergence Theorem,

Show this on the board!

And you get Gauss's Law from the 1st year course.

Curl of the E-field

- We have an expression for E.
- We can 'just' take it's curl…

Board Work

$$\nabla \times \vec{E} = \frac{1}{4\pi\varepsilon_0} \nabla \times \int_{all\,space} \frac{\vec{r} - \vec{r}'}{\left|\vec{r} - \vec{r}'\right|^3} \rho(\vec{r}')dV'$$

Does not look quite so nice.
We ***can*** try to use the fact that the curl is with respect to the unprimed coordinates again.

But maybe it would be better to start this one from the end and work backwards.

Curl of the E-field

$$\int_S \left(\nabla \times \vec{E}\right) \cdot d\vec{a} = 0$$

But we have an arbitrary Surface.

So it must be the integrand which is zero!

Electro-Statics!

$$\nabla \times \vec{E} = 0$$

Important to Remember This

Curl of the E-field

$$\nabla \times \vec{E} = 0$$

This is pretty Important for the 2nd year course
Because the static E-field has no curl
And Since:

$$\nabla \times \nabla V(\vec{r}) = 0$$

We can <u>define</u> ***E*** = *-grad(V)*

Recall
E-field from Continuous charge

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int_{all\,space} \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \rho(x', y', z') dx'dy'dz'$$

and $$\vec{E} = -\nabla V$$

but $$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Which, again, has nothing to do with the Primed coordinates!

Form of Scalar Potential

$$-\nabla V = \frac{-1}{4\pi\varepsilon_0} \int_{all\,space} \rho(\vec{r}\,')\nabla\left(f(\vec{r})\right)dV'$$

Apparently then:

$$\nabla\left[f(\vec{r})\right] = \frac{\vec{r}-\vec{r}\,'}{\left|\vec{r}-\vec{r}\,'\right|^3}$$

Great if we can find out what *f(***r***)* is! Where the origin is though must be arbitrary. So I am going to set the origin at the location of charge element.

$$-\nabla f(\vec{r}) = \frac{\vec{r}}{\left|\vec{r}\right|^3} = \frac{\hat{r}}{\vec{r}^{\,2}}$$

The result is in spherical coordinates

$$\nabla f = \hat{r}\frac{\partial f}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial f}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

So can we get a clue from the grad. in spherical coordinates?

Form of Scalar Potential

The result does not depend on Theta or Phi.
So *f* cannot be a function of either of those.

Apparently then: *f* = *f(r)*
ONLY

$$-\nabla f = -\hat{r}\frac{\partial f}{\partial r} = \frac{\hat{r}}{\vec{r}^{\,2}}$$

$$-\frac{\partial f}{\partial r} = 1/r^2$$

$$\frac{\partial f}{\partial r} = -1/r^2 \quad \Rightarrow \quad \boxed{f(r) = 1/r}$$

Form of Scalar Potential

So we discover that:

$$-\nabla V = \frac{-1}{4\pi\varepsilon_0} \int_{all\,space} \rho\left(\vec{r}\,'\right) \nabla \left(1 \Big/ \left|\vec{r} - \vec{r}\,'\right| \right) dV'$$

Remember that the 'del' operator did not operate on the primed coorindates. So we find the expression for 'V' alone.

$$V = \frac{1}{4\pi\varepsilon_0} \int_{all\,space} \left(\rho\left(\vec{r}\,'\right) \Big/ \left|\vec{r} - \vec{r}\,'\right| \right) dV'$$

Clean up final bits.

From Gauss's Law:

$$\nabla \cdot \vec{E} = \rho(\vec{r}) / \varepsilon_0$$

$$\nabla \cdot (-\nabla V) = \rho(\vec{r}) / \varepsilon_0$$

$$\nabla^2 V = -\rho(\vec{r}) / \varepsilon_0$$

Poisson's equation!!

If there happens to be no charge in a region then we get the Laplace Equation:

$$\nabla^2 V = 0$$