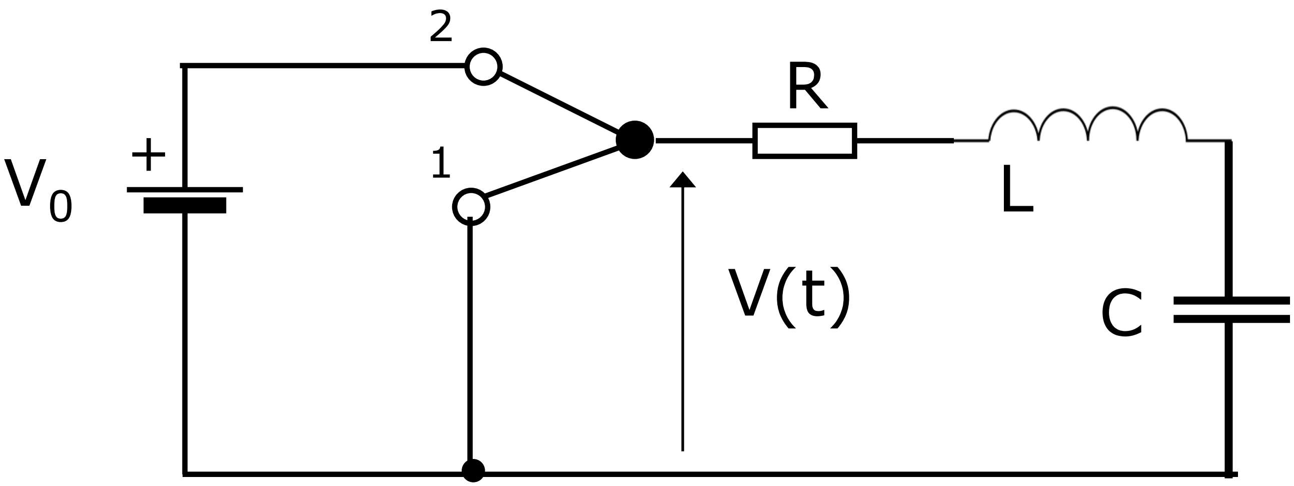
LCR circuit

$$V(t) = \begin{cases} 0 & \text{for } t < 0 \\ V_0 & t \geq 0 \end{cases}$$



$$I(t) = 0 \text{ for } t < 0$$

$$t \geq 0$$

$$V_R + V_L + V_C = V_0$$

$$IR + L\frac{dI}{dt} + \frac{Q}{C} = V_0$$

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = 0$$

$$I = ke^{\lambda t}$$ trial solution

$$\lambda^2 + \lambda\tfrac{R}{L} + \tfrac{1}{LC} = 0$$ characteristic equation

$$\lambda = -\tfrac{R}{2L} \pm \sqrt{\left(\tfrac{R}{2L}\right)^2 - \tfrac{1}{LC}}$$

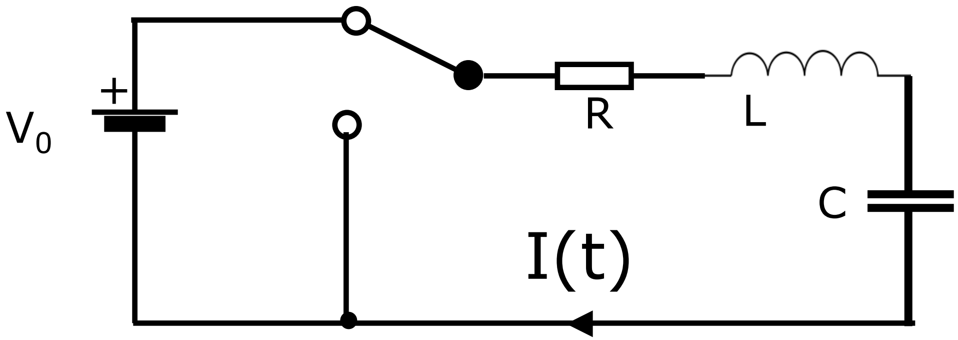
$$= -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

$$\gamma = \tfrac{R}{2L}$$

$$\omega_0^2 = \tfrac{1}{LC}$$

$$I = k_1e^{\lambda_1 t} + k_2e^{\lambda_2 t}$$ general solution

LCR circuit – initial conditions



Initial conditions $I(0),\ \frac{dI}{dt}(0)$?

No energy stored $W_L = \frac{1}{2}LI^2 = 0 \rightarrow I(0) = 0$

$$W_C = \frac{1}{2}CV^2 = 0 \rightarrow V_C(0) = 0$$

$$I(0)R + L\frac{dI}{dt}(0) + V_C = V_0$$

$$L\frac{dI}{dt}(0) = V_0$$

$$\frac{dI}{dt}(0) = \frac{V_0}{L}$$

LCR circuit

$$I = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} \qquad \lambda = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

$\gamma^2 > \omega_0^2$

$$\lambda_{1,2} = -\gamma \pm \beta \qquad \beta = \sqrt{\gamma^2 - \omega_0^2}$$

$$\gamma = \tfrac{R}{2L}$$

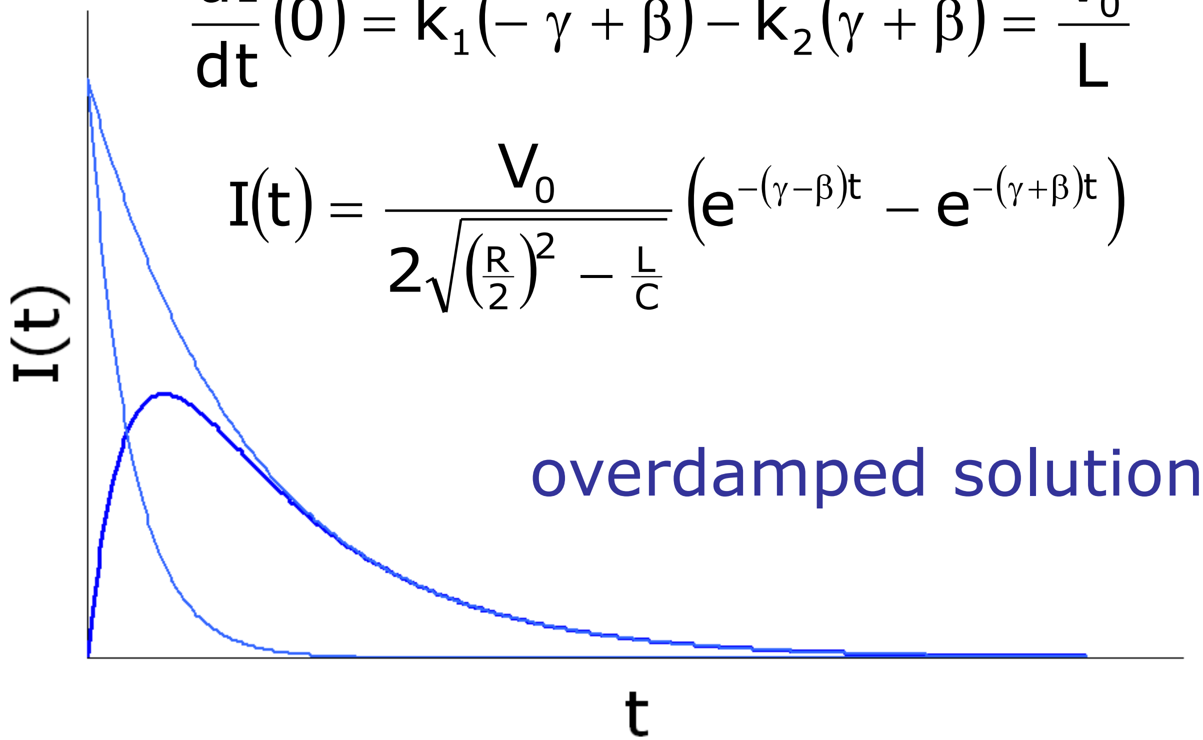
$$\omega_0^2 = \tfrac{1}{LC}$$

$$I(t \geq 0) = k_1 e^{(-\gamma+\beta)t} + k_2 e^{(-\gamma-\beta)t}$$

$$I(0) = k_1 + k_2 = 0 \rightarrow k_1 = -k_2$$

$$\frac{dI}{dt}(0) = k_1(-\gamma + \beta) - k_2(\gamma + \beta) = \frac{V_0}{L} \qquad k_1 = \frac{V_0}{2\beta L}$$

$$I(t) = \frac{V_0}{2\sqrt{\left(\tfrac{R}{2}\right)^2 - \tfrac{L}{C}}}\left(e^{-(\gamma-\beta)t} - e^{-(\gamma+\beta)t}\right)$$



overdamped solution

LCR circuit

$$\lambda = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

$$\gamma^2 = \omega_0^2$$

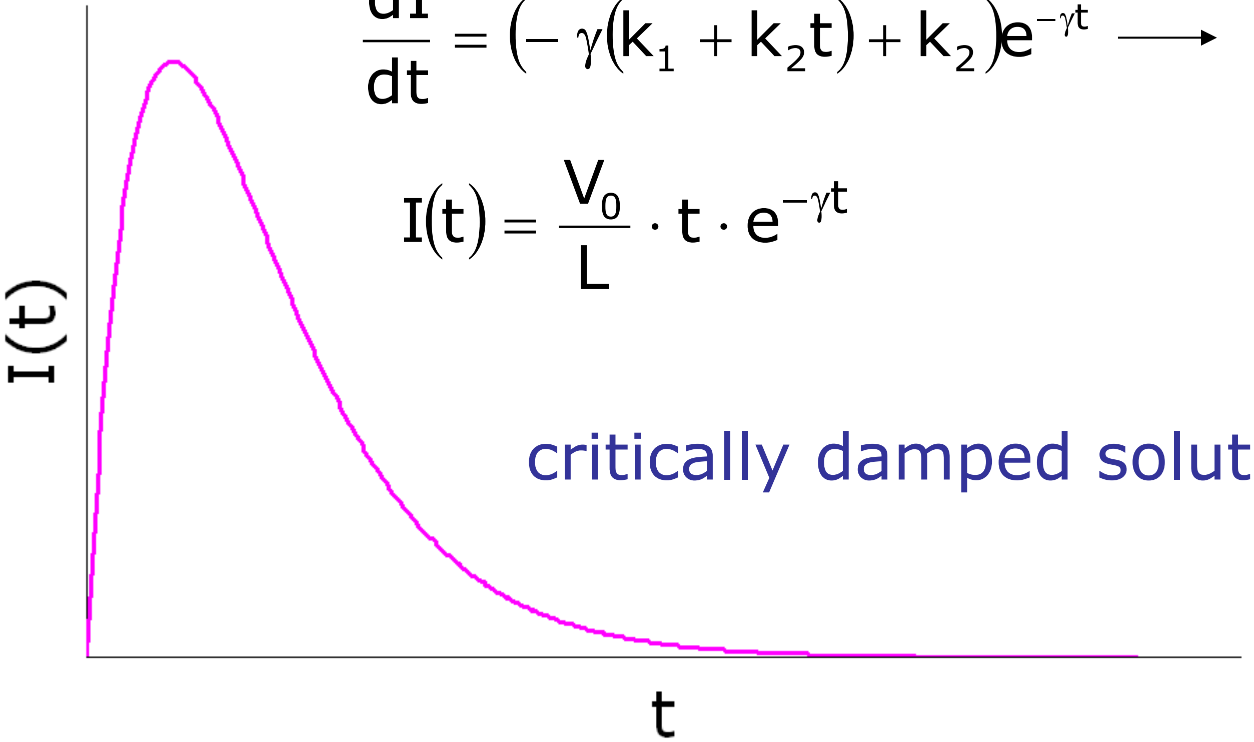
$$\lambda_1 = \lambda_2 = -\gamma$$

$$I = (k_1 + k_2 t)e^{\lambda t}$$

$$I(0) = 0 \rightarrow k_1 = 0$$

$$\frac{dI}{dt} = \left(-\gamma(k_1 + k_2 t) + k_2\right)e^{-\gamma t} \longrightarrow k_2 = \left.\frac{dI}{dt}\right|_{t=0} = \frac{V_0}{L}$$

$$I(t) = \frac{V_0}{L} \cdot t \cdot e^{-\gamma t}$$

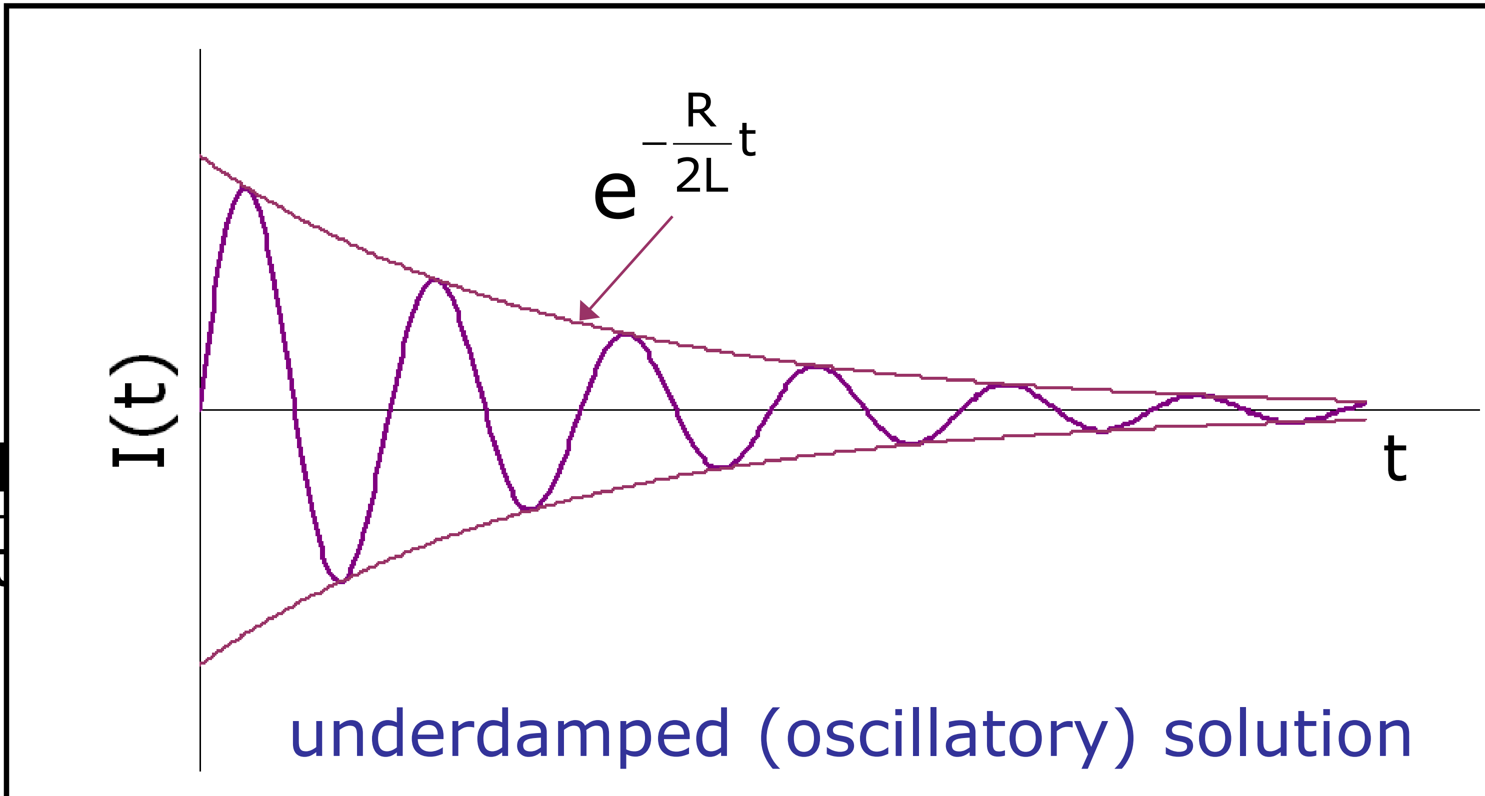


critically damped solution

LCR circuit

$$I = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} \qquad \lambda = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

$\gamma^2 < \omega_0^2$

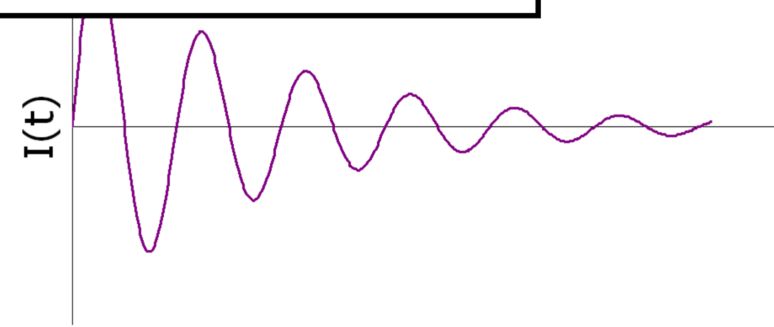


$\frac{dI}{dt}$

$\frac{V_0}{\ldots j\omega_R L}$

$I(t \geq$

$$I(t \geq 0) = \frac{V_0}{\sqrt{\frac{L}{C} - \left(\frac{R}{2}\right)^2}} e^{-\gamma t} \cdot \sin(\omega_R t)$$



LCR circuit

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = 0$$

$$I = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} \qquad \lambda = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

$$\gamma = \frac{R}{2L}$$

$$\omega_0^2 = \frac{1}{LC}$$

overdamped solution

$$\gamma^2 > \omega_0^2 \qquad \lambda_{1,2} = -\gamma \pm \beta$$

β is real

critically damped solution

$$\gamma^2 = \omega_0^2 \qquad \lambda_1 = \lambda_2 = -\gamma$$

$$I = (k_1 + k_2 t)e^{\lambda t}$$

underdamped (oscillatory) solution

$$\gamma^2 < \omega_0^2$$

$$\lambda_{1,2} = -\gamma \pm j\sqrt{\omega_0^2 - \gamma^2}$$

$$= -\gamma \pm j\omega_R$$

LCR circuit

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = 0$$

$$\gamma = \frac{R}{2L}$$

$$I = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} \qquad \lambda = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

$$\omega_0^2 = \frac{1}{LC}$$

overdamped solution

$$\gamma^2 > \omega_0^2 \qquad I(t) = \frac{V_0}{2\sqrt{\left(\frac{R}{2}\right)^2 - \frac{L}{C}}}\left(e^{-(\gamma-\beta)t} - e^{-(\gamma+\beta)t}\right)$$

critically damped solution

$$\gamma^2 = \omega_0^2 \qquad I(t) = \frac{V_0}{L} \cdot t \cdot e^{-\gamma t}$$

underdamped (oscillatory) solution

$$\gamma^2 < \omega_0^2 \qquad I(t \geq 0) = \frac{V_0}{\sqrt{\frac{L}{C} - \left(\frac{R}{2}\right)^2}} e^{-\gamma t} \cdot \sin(\omega_R t)$$

CP2 September 2003

7. Consider a capacitor C, which has an initial charge Q_0, in series with an inductor L, a resistance R and a switch. At time $t = 0$ the switch is closed and the capacitor starts to discharge through L and R. Show that the differential equation for the charge Q on the capacitor can be written as

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}t^2} + 2\alpha\frac{\mathrm{d}Q}{\mathrm{d}t} + \omega_0^2 Q = 0.$$

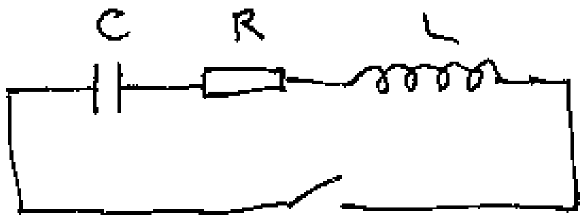
Give expressions for α and ω_0 in terms of L, C and R. [6]

By writing $Q = qe^{-\alpha t}$, where q is a function of t, show that

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + (\omega_0^2 - \alpha^2)q = 0,$$

and give the general solution for Q when $\omega_0^2 = \alpha^2$. [5]

For $L = 10\,\mathrm{mH}$, $R = 200\,\Omega$ and $\omega_0^2 = \alpha^2$, find at what time the current reaches its maximum value after the switch is closed. [9]



$$V_C + V_R + V_L = 0$$

$$\frac{Q}{C} + IR + L\frac{dI}{dt} = 0 \qquad \text{but } I = \frac{dQ}{dt}$$

so

$$\frac{Q}{C} + \frac{dQ}{dt}R + L\frac{d^2Q}{dt^2} = 0$$

$$\frac{d^2Q}{dt^2} + \underbrace{\frac{R}{L}}_{2\alpha}\frac{dQ}{dt} + \underbrace{\frac{1}{LC}}_{\omega_0^2}Q = 0$$

Try solution $Q = q e^{-\alpha t}$

$$\dot{Q} = \dot{q} e^{-\alpha t} - \alpha q e^{-\alpha t}$$

$$\ddot{Q} = \ddot{q} e^{-\alpha t} - \alpha \dot{q} e^{-\alpha t} - \alpha \dot{q} e^{-\alpha t} + \alpha^2 q e^{-\alpha t}$$

$$= \ddot{q} e^{-\alpha t} - 2\alpha \dot{q} e^{-\alpha t} + \alpha^2 q e^{-\alpha t}$$

$$\underbrace{(\ddot{q} - 2\alpha \dot{q} + \alpha^2 q) e^{-\alpha t}}_{\frac{d^2Q}{dt^2}} + 2\alpha \underbrace{(\dot{q} - \alpha q) e^{-\alpha t}}_{\frac{dQ}{dt}} + \omega_0^2 \underbrace{q e^{-\alpha t}}_{Q} = 0$$

$$\rightarrow \left[\ddot{q} + (\omega_0^2 - \alpha^2) q\right] e^{-\alpha t} = 0$$

$$\rightarrow \ddot{q} + (\omega_0^2 - \alpha^2) q = 0$$

Solve $\ddot{q} + (\omega_0^2 - \alpha^2) q = 0$ when $\omega_0^2 = \alpha^2$

$$\ddot{q} = 0$$
$$\dot{q} = A$$
$$q = At + B$$

so $Q = (At + B) e^{-\alpha t}$

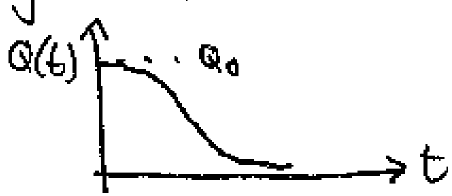
$\dot{Q} = (A - \alpha A t - \alpha B) e^{-\alpha t}$

at $t = 0$ $Q = Q_0$ $\dot{Q} = 0$ (thanks to inductor)

$\rightarrow B = Q_0$ $A = \alpha Q_0$
$= \omega_0 Q_0$

$$Q(t) = Q_0 (\omega_0 t + 1) e^{-\omega_0 t}$$

"Critically damped" solution



[Recall 2 other solutions

- $\alpha^2 > \omega_0^2$ overdamped - slower return to baseline than above
- $\alpha^2 < \omega_0^2$ underdamped - oscillatory solution within exponential fall-off]

Maximum current $L = 10\,mH$ $R = 200\,\Omega$

$$I = \dot{Q} = -\omega_0^2 Q_0 t e^{-\omega_0 t}$$

$$\dot{I} = -\omega_0^2 Q_0 e^{-\omega_0 t} + \omega_0^3 Q_0 t e^{-\omega_0 t}$$

$$= 0 \text{ when } t = 1/\omega_0$$

$$\omega_0 = \alpha = \frac{R}{2L} = \frac{200}{2 \times 10 \times 10^{-3}} = 10^4 s^{-1}$$

$\rightarrow$ I maximal at $t = 0.1\,ms$

LCR circuit

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = 0$$

$$\gamma = \tfrac{R}{2L}$$

$$I = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} \qquad \lambda = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

$$\omega_0^2 = \tfrac{1}{LC}$$

overdamped solution

$$\gamma^2 > \omega_0^2 \qquad I(t) = \frac{V_0}{2\sqrt{\left(\frac{R}{2}\right)^2 - \frac{L}{C}}}\left(e^{-(\gamma-\beta)t} - e^{-(\gamma+\beta)t}\right)$$

critically damped solution

$$\gamma^2 = \omega_0^2 \qquad I(t) = \frac{V_0}{L}\cdot t \cdot e^{-\gamma t}$$

underdamped (oscillatory) solution

$$\gamma^2 < \omega_0^2 \qquad I(t \geq 0) = \frac{V_0}{\sqrt{\frac{L}{C} - \left(\frac{R}{2}\right)^2}} e^{-\gamma t}\cdot \sin(\omega_R t)$$