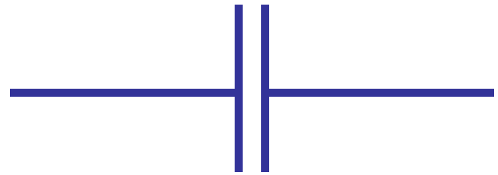
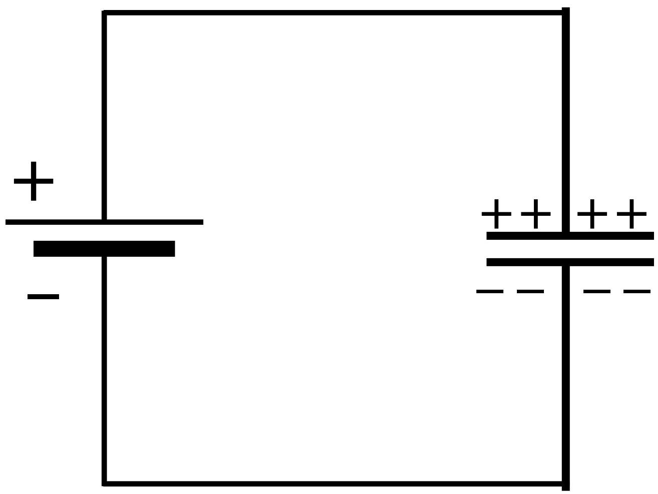
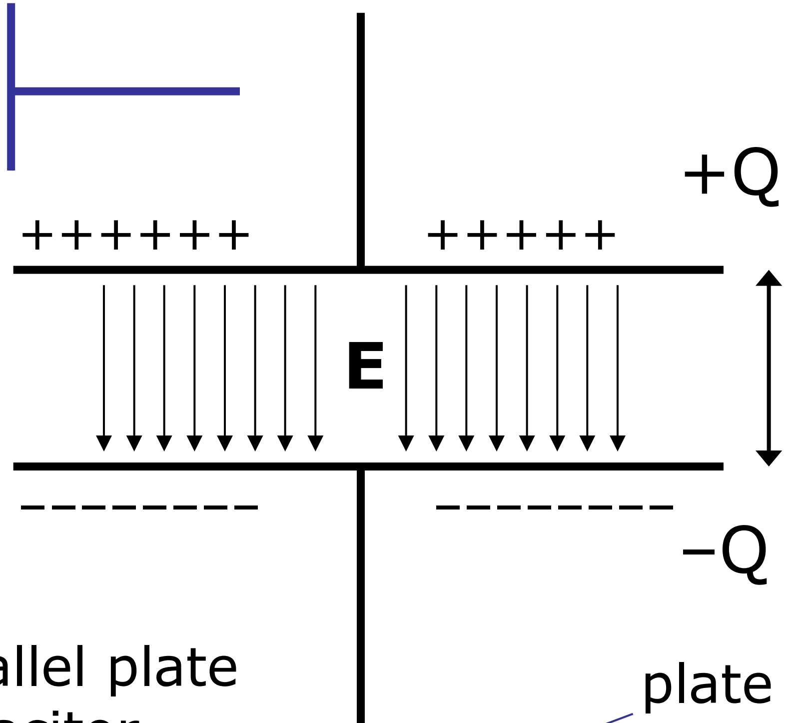
Capacitors



Capacitors store charge





Parallel plate capacitor

plate area

$$C = \frac{\varepsilon A}{d}$$

$$Q = CV \qquad E = \frac{V}{d}$$

$$I = \frac{dQ}{dt} \longrightarrow I = C\frac{dV}{dt}$$

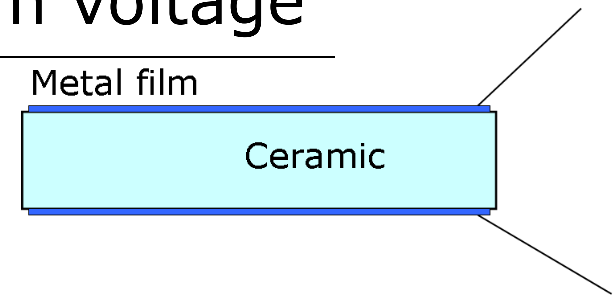
Dielectric permittivity $\varepsilon = \varepsilon_r \varepsilon_0$

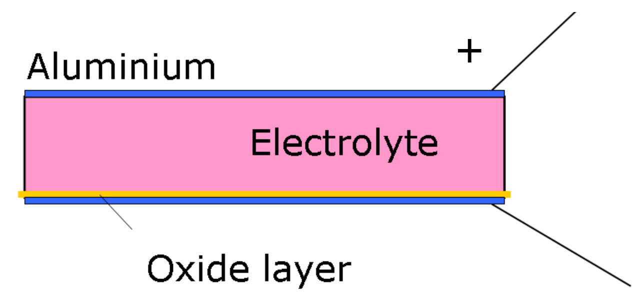
$$\varepsilon_0 = 8.854 \times 10^{-12}\, CV^{-1}m^{-1}$$

Units: farad [F]=[AsV^{-1}] more usually μF, nF, pF

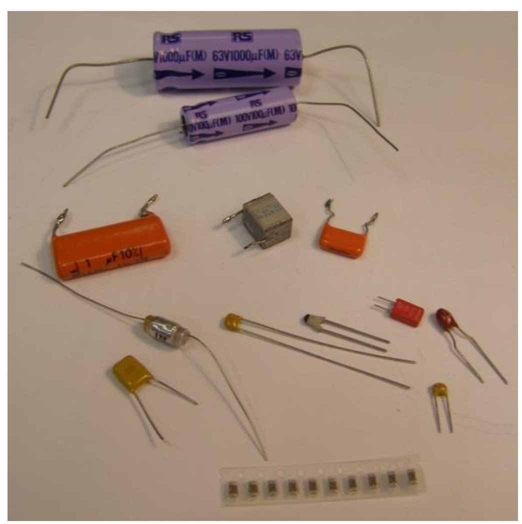
Capacitor types

	Range	Maximum voltage	
Ceramic	1pF–1μF	30kV	
Mica	1pF–10nF	500V	(very stable)
Plastic	100pF–10μF	1kV	
Electrolytic	0.1μF–0.1F	500V	
Parasitic	~pF		



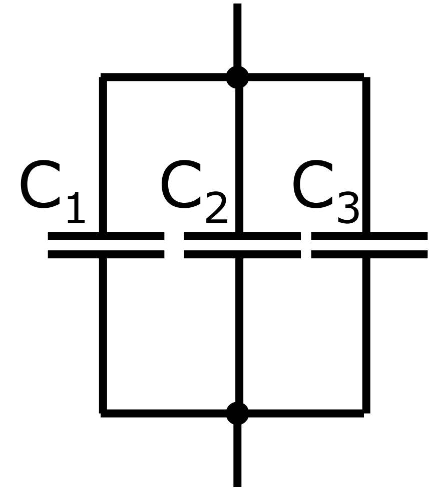


Polarised



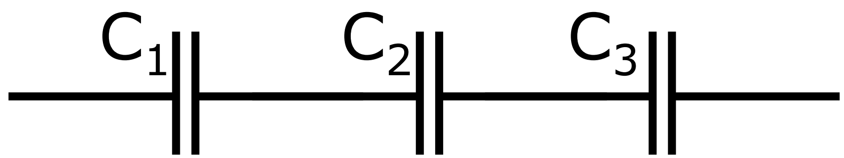
Capacitors

Capacitors in parallel:

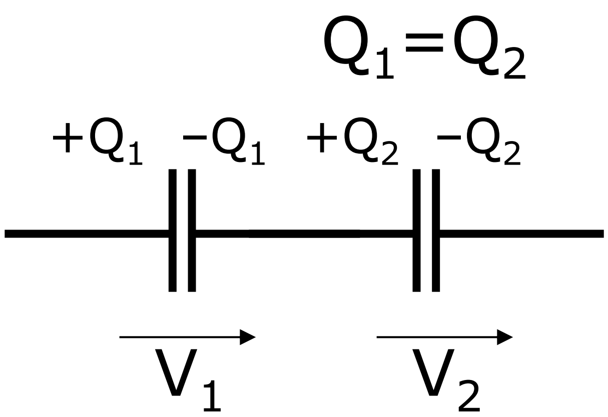


$$C_{Total} = \sum_n C_n = C_1 + C_2 + C_3 \ldots$$

Capacitors in series:



$$\frac{1}{C_T} = \sum_n \frac{1}{C_n} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \ldots$$



$$V = \frac{Q}{C_T} = \underbrace{\frac{Q_1}{C_1}}_{V_1} + \underbrace{\frac{Q_2}{C_2}}_{V_2} + \ldots$$

Capacitors – energy stored

$$W = \int_0^t P(t')dt' = \int_0^t I(t')V(t')dt' = \int_0^t C\frac{dV(t')}{dt'}V(t')dt'$$

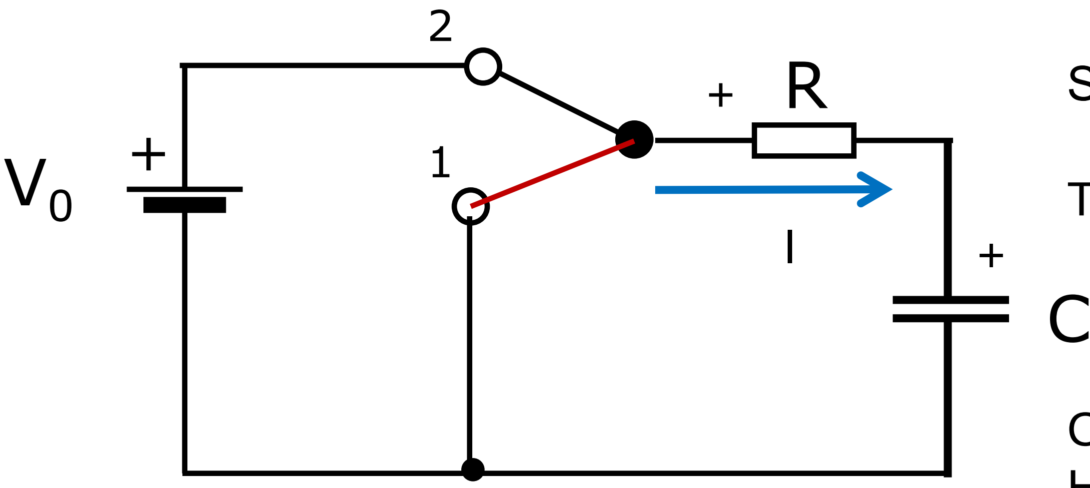
$$= C\int_0^V V'\,dV' = \tfrac{1}{2}CV^2$$

$$W = \tfrac{1}{2}CV^2 = \tfrac{1}{2}QV = \frac{Q^2}{2C}$$

Energy stored as electric field

RC circuits

1 Initially $V_R=0$ $V_C=0$



Switch in Position “1” for a long time.

Then instantly flips at time t = 0.

Capacitor has a derivative!
How do we analyse this?

There are some tricks…

ALWAYS start by asking these three questions:

1.) What does the Circuit do up until the switch flips?
(Switch has been at pos. 1 for a VERY long time. easy)

2). What does the circuit do a VERY long time AFTER the switch flips? (easy)

3). What can we say about the INSTANT after switch flips? (easy if you know trick)

The Trick!!

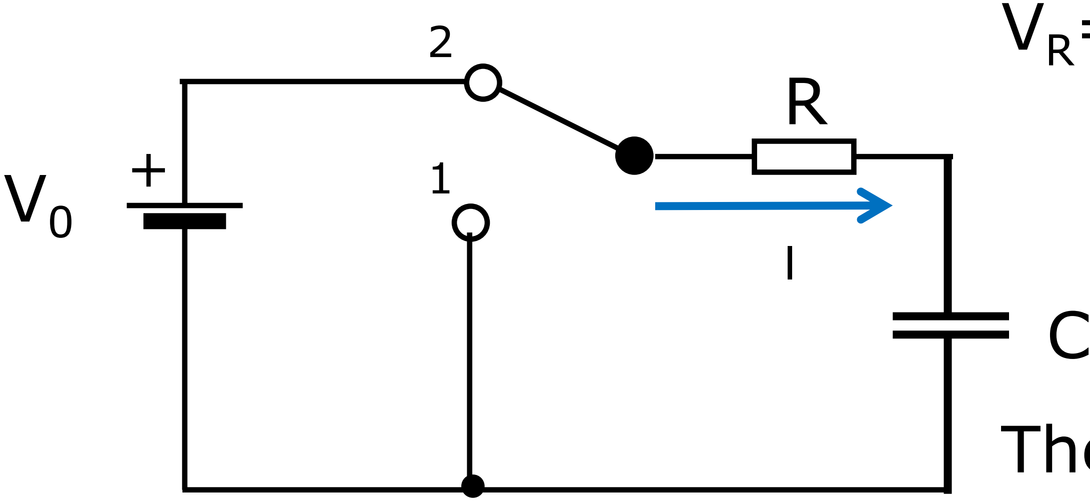
- Remember → $I = C\frac{dV}{dt}$
- Suppose the voltage on a 1 farad Capacitor changes by 1 volt in 1 second.
  - What is the current?
  - What if the same change in V happens in 1 microsecond?
  - So…What if the same change in V happens instantly?
- Rule: It is impossible to change the voltage on a capacitor instantly!
  - Another way to say it:
    The voltage at t = 0-ε is the same as at t = 0+ε.

Note for Todd : Put problem on board and work through these points!

RC circuits

1 Initially at $t = 0^-$

$V_R=0$ $V_C=0$ $I=0$



Then at $t = 0^+$

Must be that $V_C=0$

If $V_C=0$ then KVL says $V_R=V_0$ and $I=V_0/R$.

Capacitor is acting like a short-circuit.

Finally at $t = +\infty$

$V_R=0$ $V_C=V_0$ $I=0$

2

$$-V_0 + V_R + V_C = 0$$

$$V_0 = V_R + V_C$$

$$= IR + \frac{Q}{C}$$

differentiate wrt t

$$0 = R\frac{dI}{dt} + \frac{I}{C}$$

$$0 = R\frac{dI}{dt} + \frac{I}{C}$$

$$\frac{-I}{RC} = \frac{dI}{dt}$$

$$\int \frac{-1}{RC}\,dt = \int \frac{1}{I}\,dI$$

$$\frac{-t}{RC} = \ln I + a$$

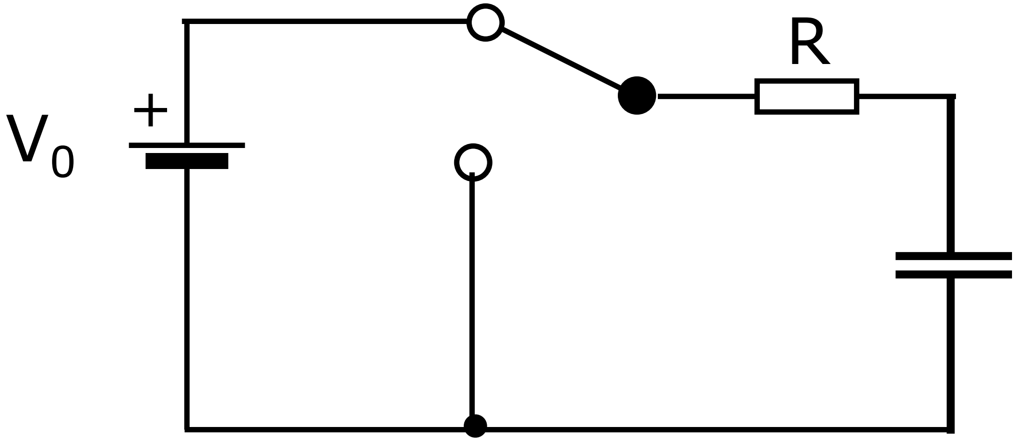
$$= \ln I + \ln b$$

$$= \ln(bI)$$

$$e^{\left(\frac{-t}{RC}\right)} = bI$$

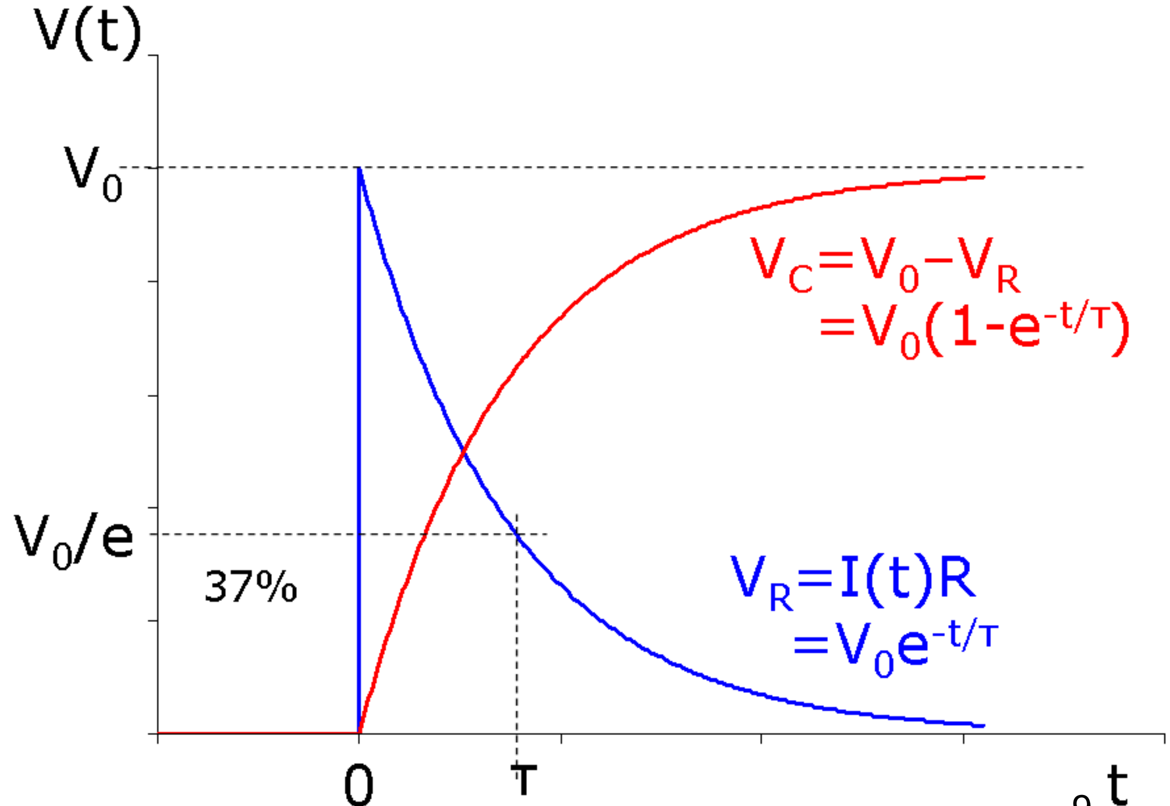
$$I = \left(\frac{1}{b}\right)e^{\left(\frac{-t}{RC}\right)}$$

$$= I_0 e^{\left(\frac{-t}{RC}\right)}$$



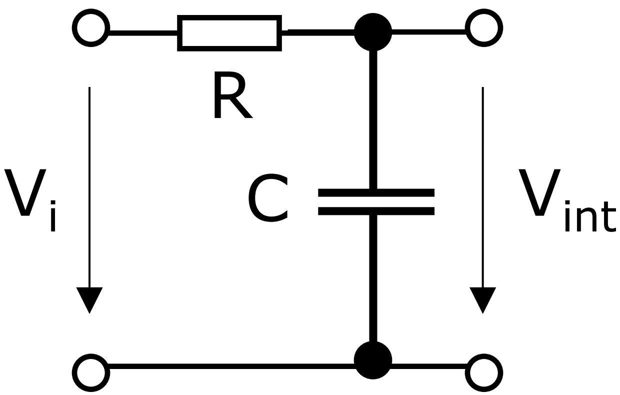
$$I(t) = \frac{V_0}{R} e^{-t/\tau}$$

$$\tau = RC$$



Integrator

Capacitor as integrator



$$V_{int} = V_C = \frac{1}{C}\int_0^t I dt'$$

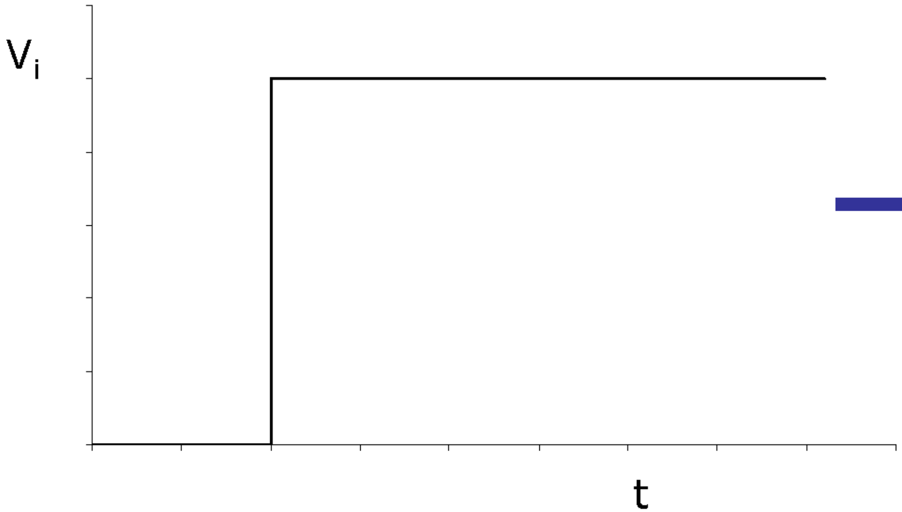
$$V_C = \frac{1}{C}\int_0^t \frac{V_i - V_C}{R} dt'$$

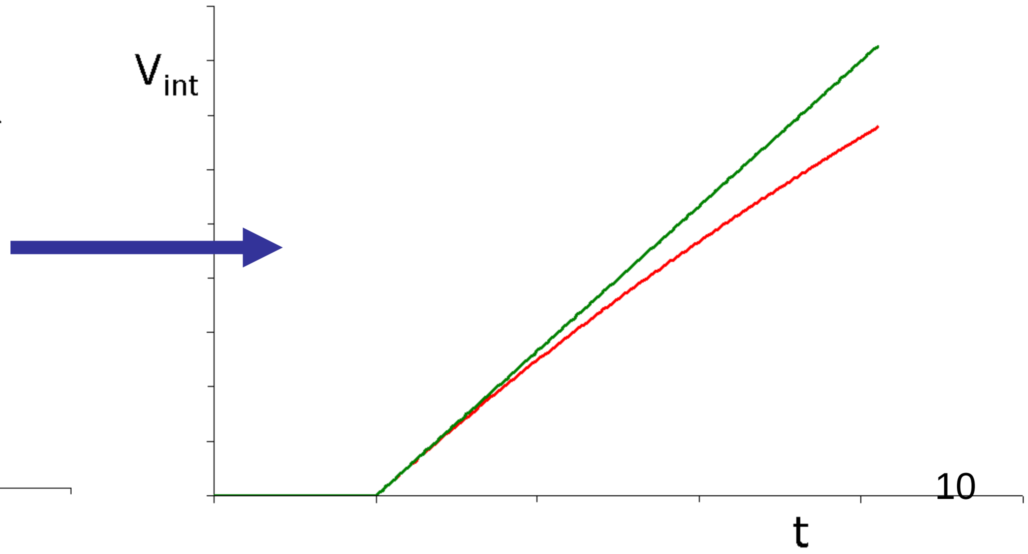
$$\approx \frac{1}{RC}\int_0^t V_i dt'$$

$I = V_i - V_c / R$

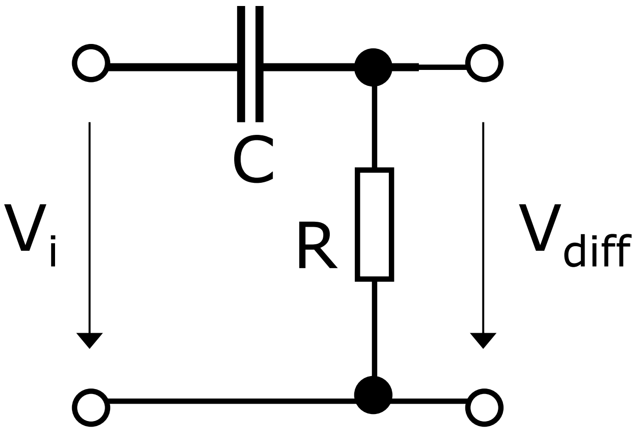
If $RC >> t$

$V_C << V_i$





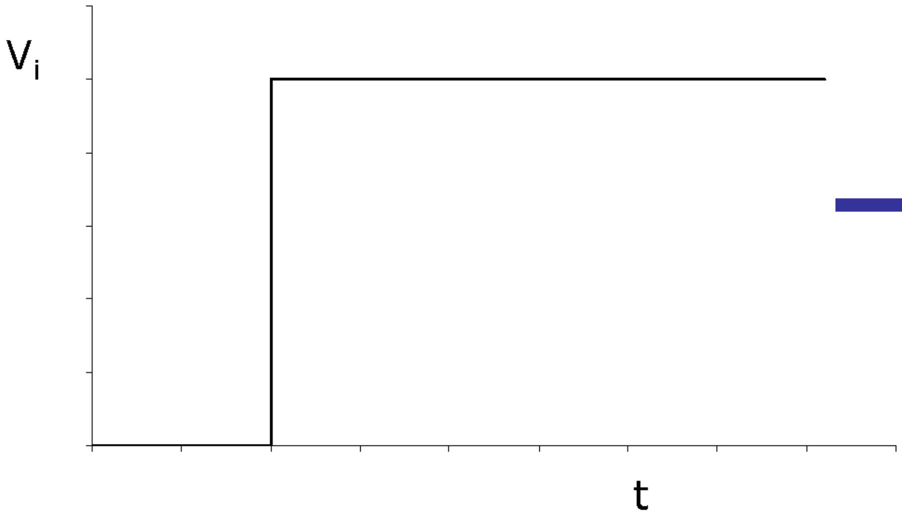
Differentiation

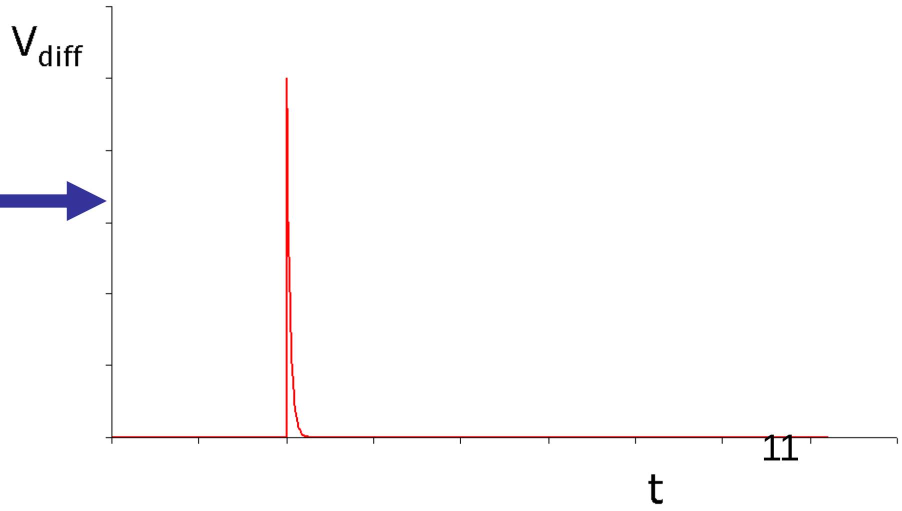


$$V_{diff} = V_R = RI = RC\left(\frac{dV_i}{dt} - \frac{dV_R}{dt}\right)$$

$$\text{Small RC} \rightarrow \quad \frac{dV_i}{dt} >> \frac{dV_R}{dt}$$

$$V_{diff} \approx RC\frac{dV_i}{dt}$$

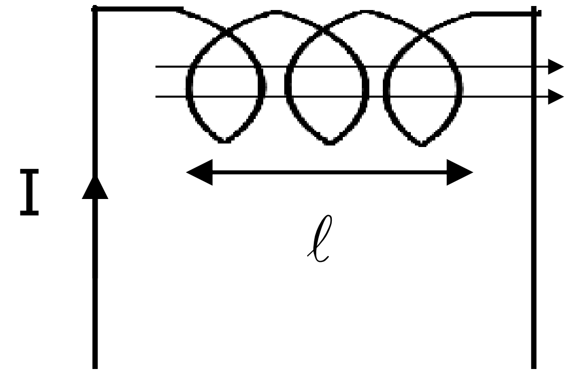




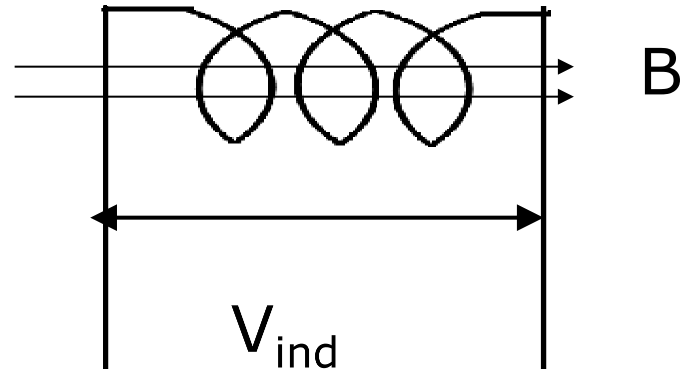
Inductors

Solenoid N turns

Electromagnetic induction



$$B = \frac{\mu_0 N I}{\ell}$$

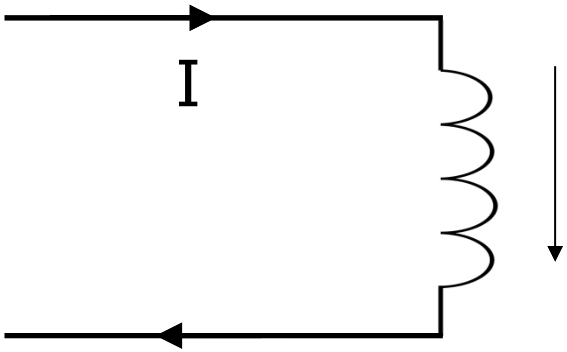


$$V_{ind} = -\underbrace{\mu_0 A \frac{N_1}{\ell} N_2}_{\text{Inductance}} \frac{dI}{dt}$$

$$V_{ind} = -\frac{d\Phi}{dt}$$

Magnetic flux $\Phi = \int B\,dA$

Self Inductance

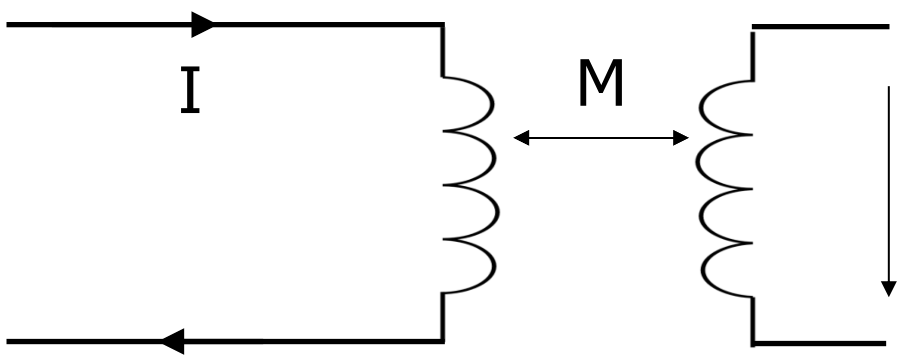


$$V = L\frac{dI}{dt}$$

$$L = \mu_0 A \frac{N^2}{\ell}$$

for solenoid

Mutual Inductance

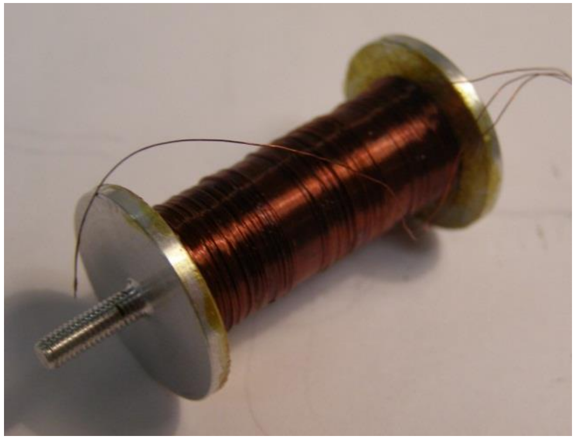


$$V = M\frac{dI}{dt}$$

Units: henrys [H] = [VsA^{-1}]

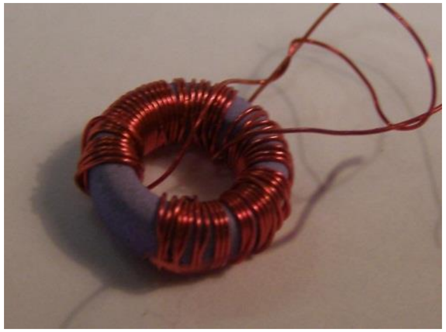
$$M = \mu_0 A \frac{N_1 N_2}{\ell}$$

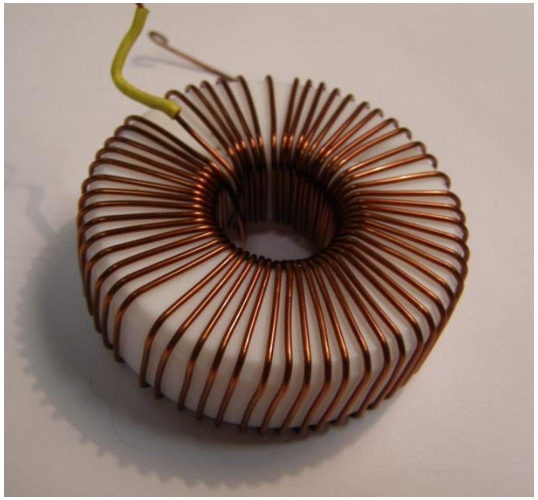
Inductors

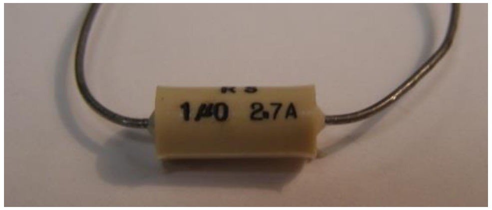


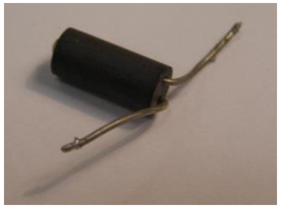
Wire wound coils - air core

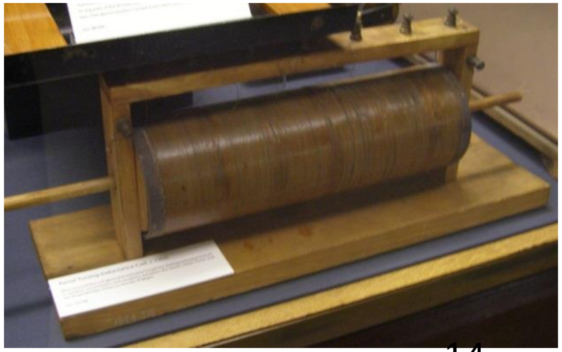
- ferrite core





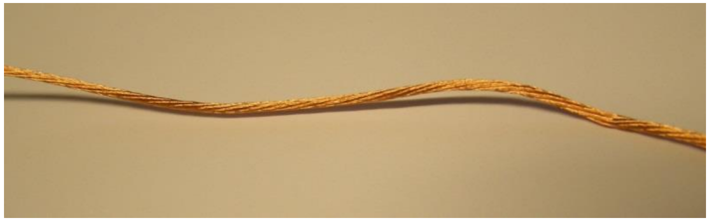




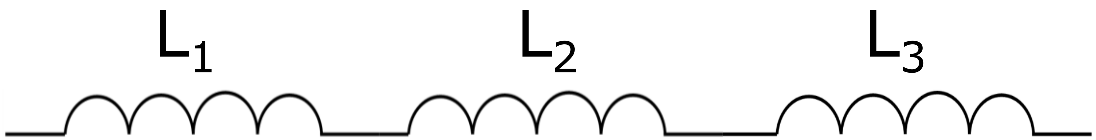


Wire loops

Straight wire



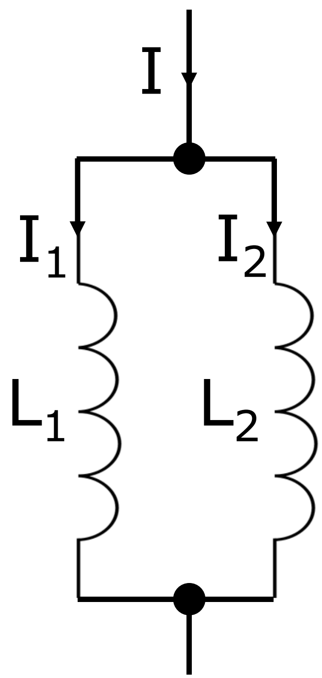
Series / parallel circuits



$$L_T = \sum_n L_n$$

$$V = L_1 \frac{dI}{dt} + L_2 \frac{dI}{dt} + L_3 \frac{dI}{dt}$$

Inductors in series: $L_{Total} = L_1 + L_2 + L_3 \ldots$



$$I = I_1 + I_2$$

$$\underbrace{\frac{dI}{dt}}_{\frac{V}{L}} = \underbrace{\frac{dI_1}{dt}}_{\frac{V}{L_1}} + \underbrace{\frac{dI_2}{dt}}_{\frac{V}{L_2}}$$

Inductors in parallel

$$\frac{1}{L_T} = \sum_n \frac{1}{L_n}$$

$$= \frac{1}{L_1} + \frac{1}{L_2} + \ldots$$

Inductors – energy stored

$$
\begin{aligned}
W &= \int_0^t P(t')\,dt' \\
&= \int_0^t I(t')V(t')\,dt' \\
&= \int_0^t I(t')L\frac{dI(t')}{dt}\,dt' \\
&= L\int_0^I I\,dI' \\
&= \tfrac{1}{2}LI^2
\end{aligned}
$$

Energy stored as magnetic field

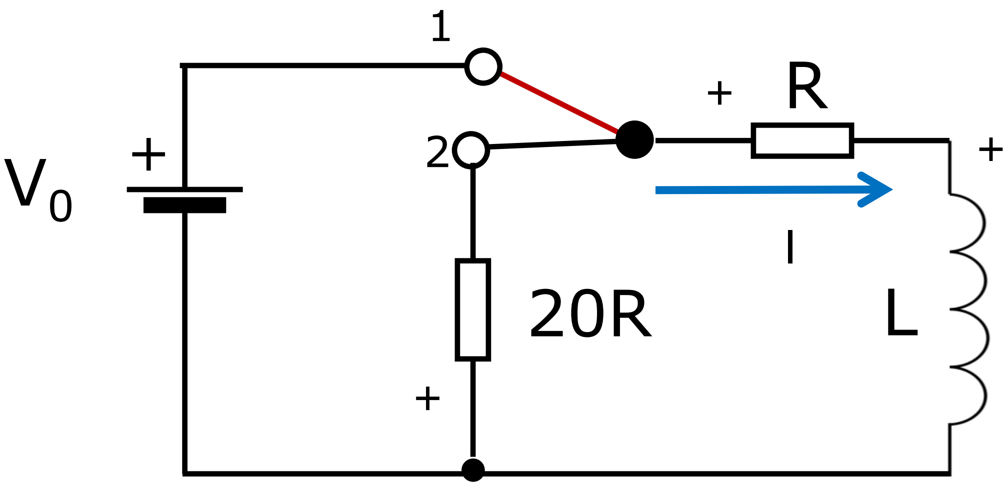
The Next Trick!!

- Remember → $V = L\frac{dI}{dt}$
- Suppose the current on a 1 henry Inductor changes by 1 amp in 1 second.
  - What is the voltage?
  - What if the same change in I happens in 1 microsecond?
  - So…What if the same change in I happens instantly?
- Rule: It is impossible to change the current on an inductor instantly!
  - Another way to say it:
    The current at $t = 0-\varepsilon$ is the same as at $t = 0+\varepsilon$.
  - Does all this seem familiar? It is the concept of "duality".

RL circuits

Initially $t=0^-$

$V_R=V_0$ $V_L=0$ $I=V_0/R$



At $t=0^+$ → Current in L MUST be the same

$I=V_0/R$ So…

$V_R=V_0$ $V_{R20}=20V_0$

and $V_L = -21V_0$!!!

For times t > 0

$$0 = V_{R20} + V_R + V_L$$

$$0 = 20IR + IR + L\frac{dI}{dt}$$

$$0 = 21IR + L\frac{dI}{dt}$$

And at $t=\infty$ (Pos. "2")

$V_R=0$ $V_L=0$ $I=0$

$$\frac{dI}{dt}+\frac{21R}{L}I=0$$

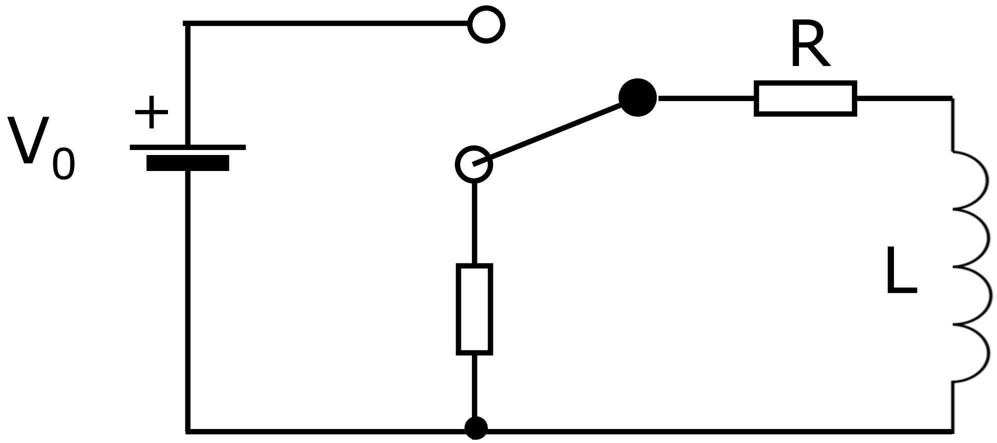
$$\int_{I(0)}^{I(t)}\frac{dI}{I}=-\int_0^t\frac{dt}{\tau} \qquad \tau=\frac{L}{21R}$$

$$\ln\left(\frac{I(t)}{I(0)}\right)=-\frac{t}{\tau}$$

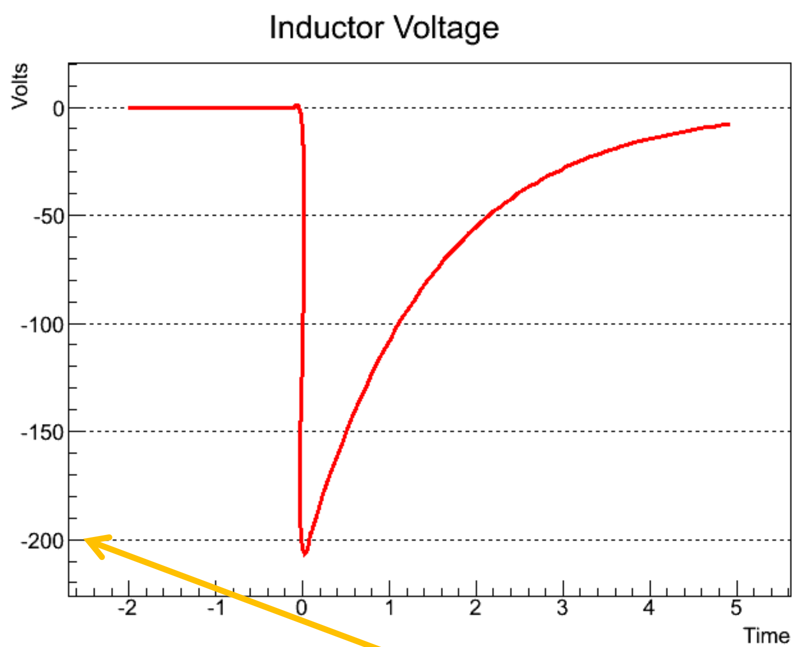
$$I(t)=I(0)\exp\left(-\frac{t}{\tau}\right)$$

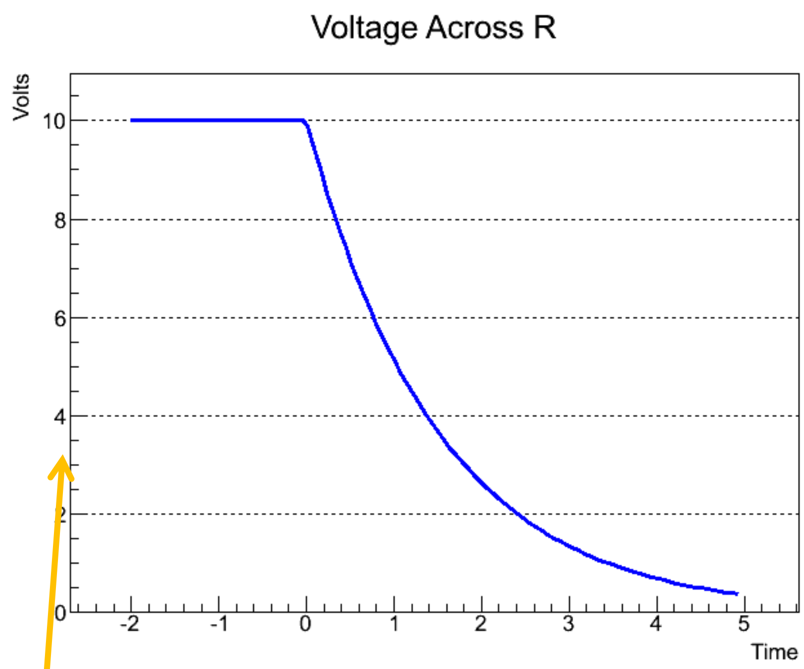
$$I(t=0)=0 \longrightarrow I(0)=\frac{V_0}{R}$$

$$I(t>0)=\frac{V_0}{R}\exp\left(-\frac{t}{\tau}\right) \qquad I(t\le 0)=\frac{V_0}{R}$$

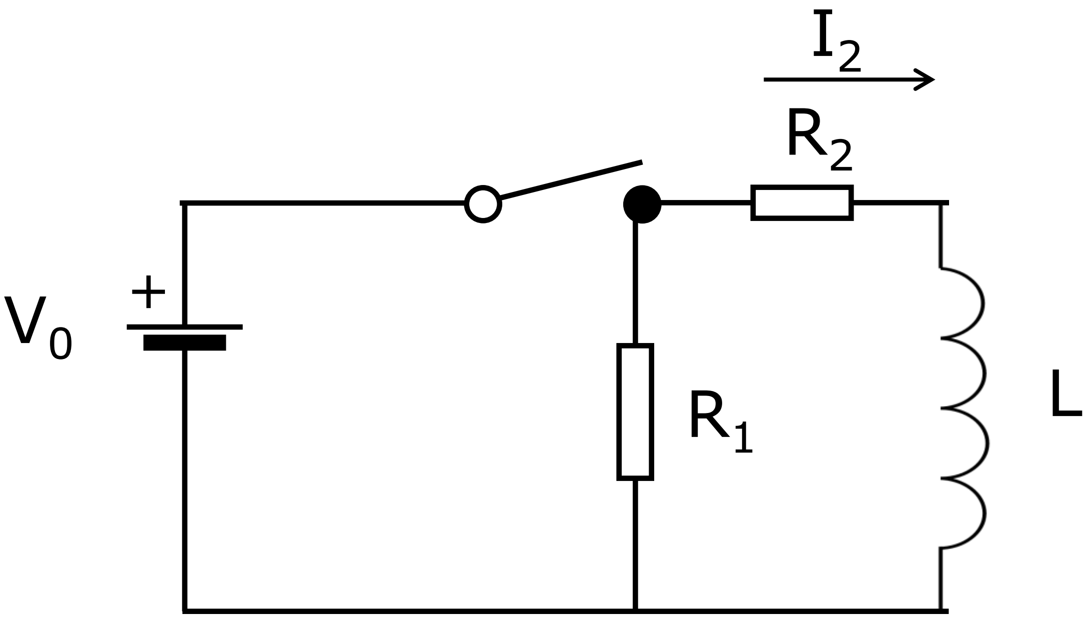


$$I(t) = \frac{V_0}{R}\left(\exp\left(-\frac{t}{\tau}\right)\right)$$





Notice Difference in Scale!



$t<0$ switch closed $I_2=?$
$t=0$ switch opened
$t>0$ $I_2(t)=?$

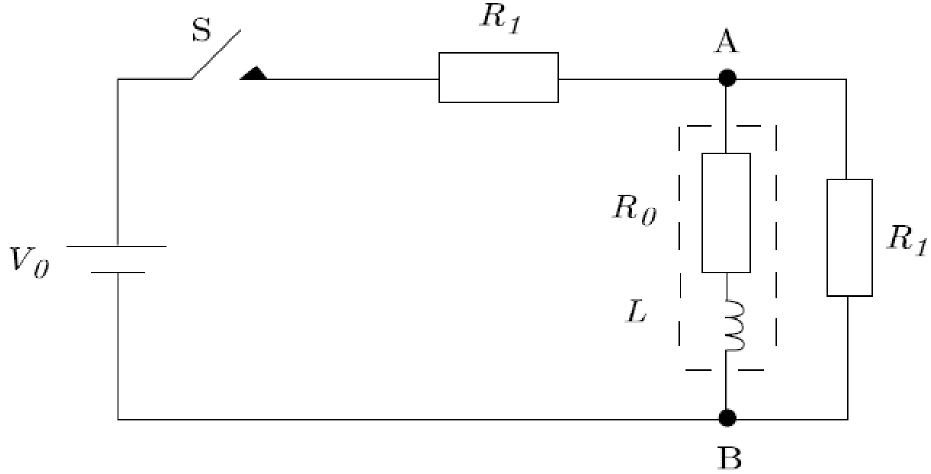
voltage across $R_1=?$
Let R_1 get very large:
What happens to V_L?

Write this problem down and DO attempt it at home

Physics 3 June 2001

Notes for Todd:
Try to do it; consult the backup slides if you run into trouble.

6.



The circuit shown contains a commercial inductor which can be represented by a resistor R_0 in series with an inductance L. Identical resistors R_1 are connected in parallel and in series with the inductor as shown in the circuit diagram. At time $t = 0$ the switch S is closed so that current flows from the battery.

(i) Show that the current, I, flowing through the inductor is given by the equation

$$\frac{V_0}{2} = \left(\frac{R_1}{2} + R_0\right) I + L\frac{dI}{dt}$$

where V_0 is the battery voltage. [11]

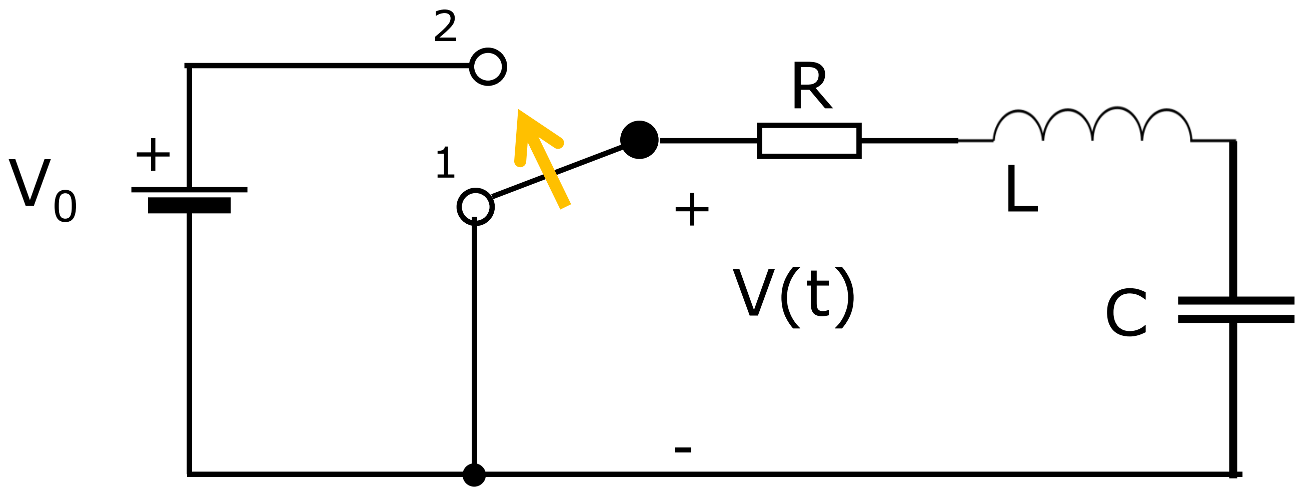
(ii) Obtain an expression for the current I as a function of time and sketch a graph of the result. [8]

(iii) Calculate the potential difference V_{AB} when

$$t = \left(\frac{2L}{2R_0 + R_1}\right).$$ [6]

LCR circuit

$$V(t) = \begin{cases} 0 & \text{for } t < 0 \\ V_0 & t \geq 0 \end{cases}$$



$I(t)=0$ for $t<0$

$$t \geq 0$$

KVL $\quad V_R + V_L + V_C = V_0$

Component Laws $\quad IR + L\frac{dI}{dt} + \frac{Q}{C} = V_0$

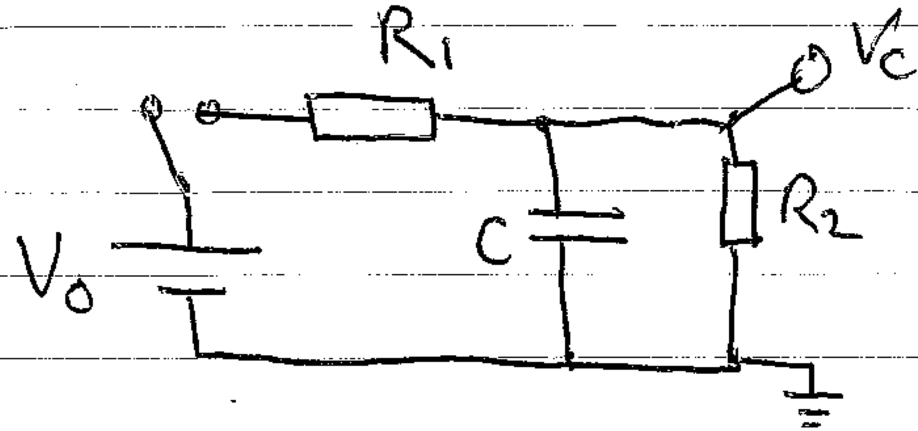
Get Rid of "Q" $\quad \frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = 0$

R, L, C circuits

- What if we have more than just one of each?
- What if we don't have switches but have some other input instead?
  - Power from the wall socket comes as a sinusoid, wouldn't it be good to solve such problems?
- **Yes! There is a much better way!**
  - **Stay Tuned!**

The Spanish Inquisition
(not unexpected)

BACKUP SLIDES



$t = 0$
$V_c = 0$

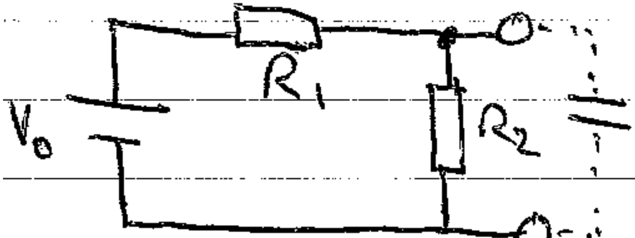
Voltage across capacitor V_c

$$I_0 = I_C + I_R$$

$$\frac{V_0 - V_c}{R_1} = C\frac{dV_c}{dt} + \frac{V_c}{R_2}$$

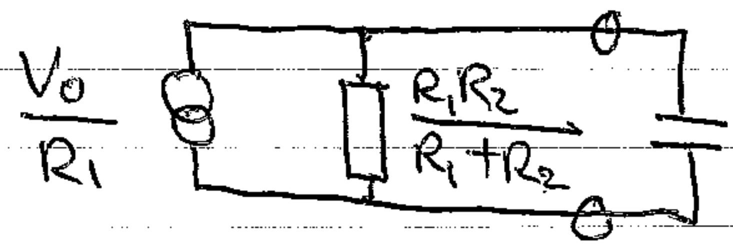
$$\frac{V_0}{R_1} = C\frac{dV_c}{dt} + V_c\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

Norton equivalent circuit



output shorted $\rightarrow$ $I_{equiv} = \dfrac{V_0}{R_1}$

voltage source shorted $\rightarrow$ $R_{equiv} = \dfrac{R_1 R_2}{R_1 + R_2}$

$\equiv$ 

$$\frac{V_0}{R_1} - \frac{V_c\,(R_1 + R_2)}{R_1 R_2} = C\,\frac{dV_c}{dt}$$

$$\int dt = \int \frac{C}{\frac{V_0}{R_1} - V_C \frac{(R_1+R_2)}{R_1 R_2}} dV_C$$

$$= \int \frac{C}{I_0 - \frac{V_C}{R_X}} dV_C \qquad R_X = \frac{R_1 R_2}{R_1 + R_2}$$

$$I_0 = \frac{V_0}{R_1}$$

$$t = -R_X C \ln\left(I_0 - \frac{V_C}{R_X}\right) + \text{const.}$$

$$\frac{-t}{R_X C} = \ln\left(I_0 - \frac{V_C}{R_X}\right) - \ln K$$

$$K e^{-t/R_X C} = I_0 - \frac{V_C}{R_X}$$

$$V_C = I_0 R_X - K e^{-t/R_X C}$$

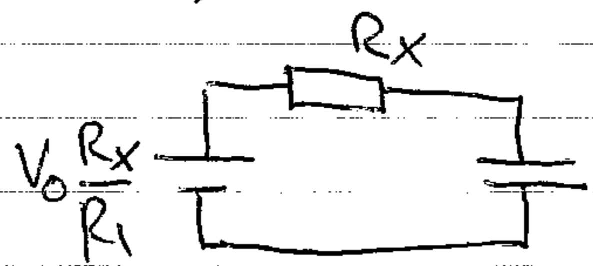
$t=0 \quad V_C = 0 \longrightarrow K = I_0 R_x$

$$V_C = I_0 R_x \left(1 - e^{-t/R_x C}\right)$$

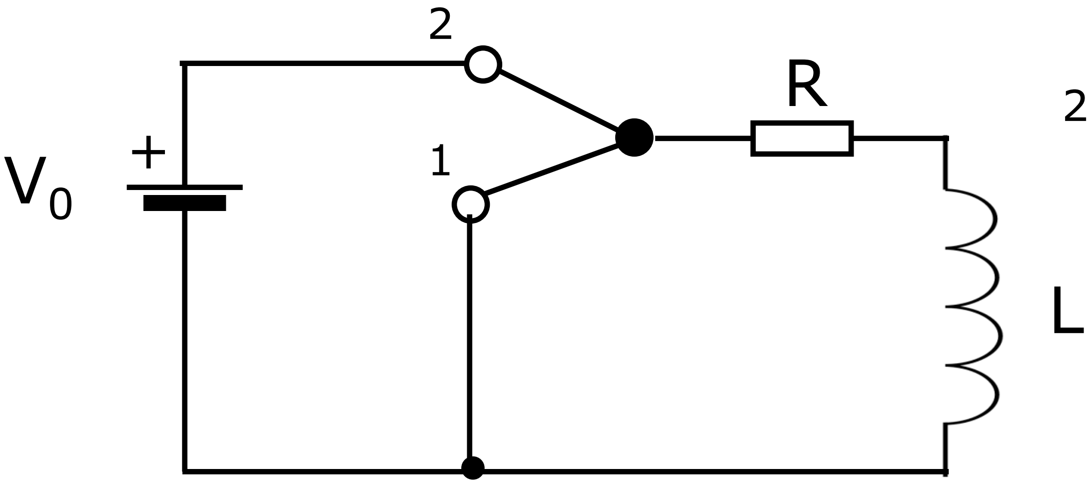
$$= V_0 \frac{R_x}{R_1}\left(1 - e^{-t/R_x C}\right)$$

$$\frac{1}{R_x} = \frac{1}{R_1} + \frac{1}{R_2}$$

Thevenin equivalent circuit



RL circuits



1 Initially $V_R=0 \quad V_L=0$

2
$$V_0 = V_R + V_L$$
$$= iR + L\frac{di}{dt}$$
$$\frac{di}{dt} + \frac{R}{L}i = \frac{V_0}{L}$$

$$\frac{di}{dt} + \frac{R}{L}i = 0$$

$$\int_{I(0)}^{I(t)} \frac{di}{i} = -\int_0^t \frac{dt}{\tau} \qquad \tau = \frac{L}{R}$$

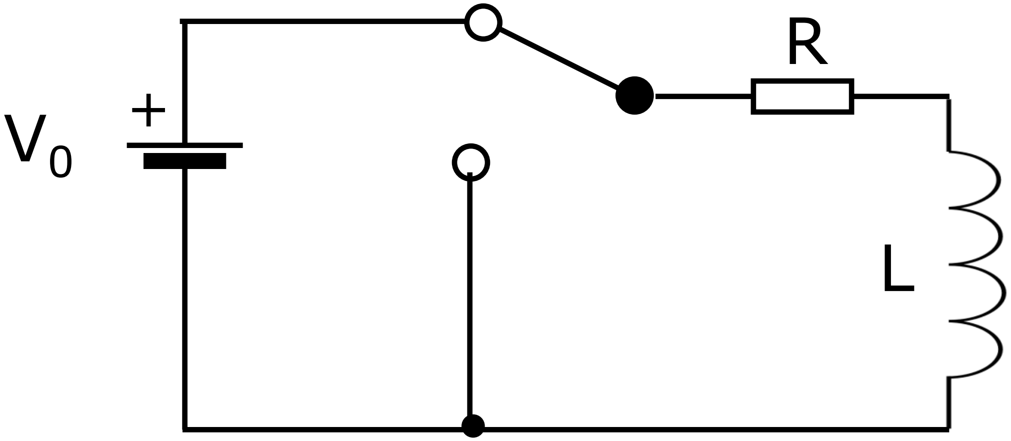
$$\ln\left(\frac{I(t)}{I(0)}\right) = -\frac{t}{\tau}$$

$$I(t) = I(0)\exp\left(-\frac{t}{\tau}\right)$$

$$I(t) = I(0)\exp\left(-\frac{t}{\tau}\right) + \frac{V_0}{R}$$

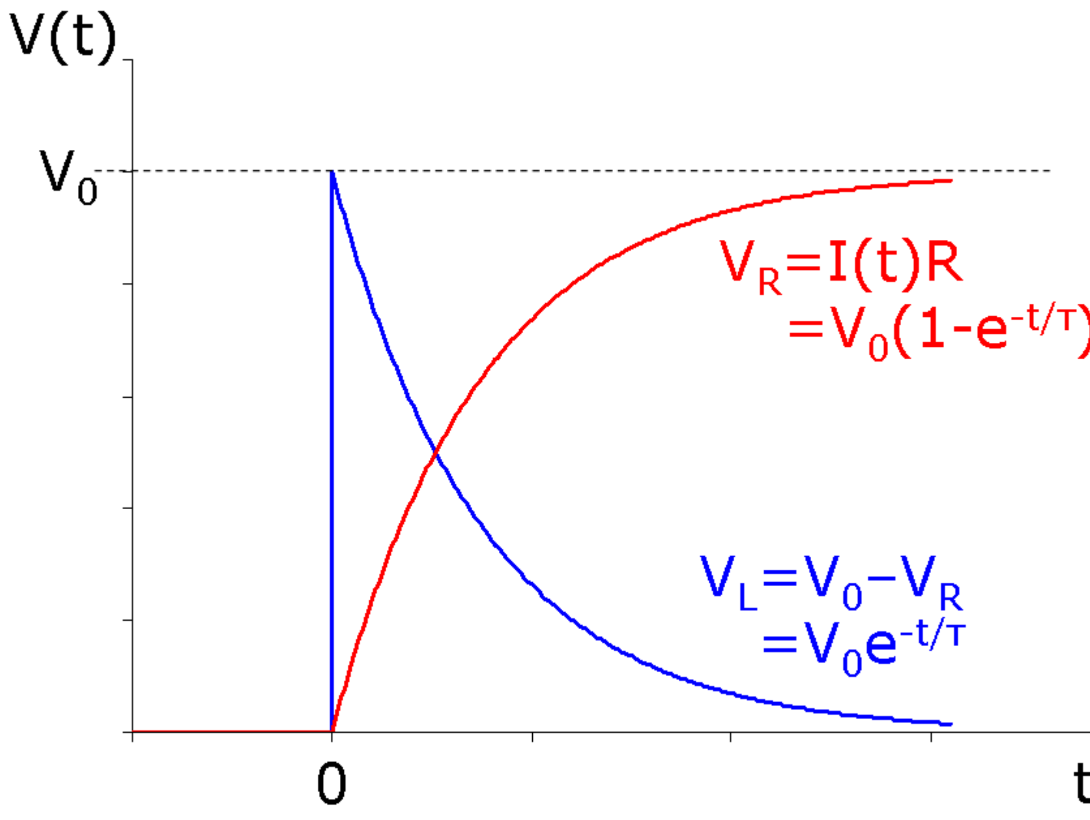
$$I(t=0) = 0 \longrightarrow I(0) = \frac{-V_0}{R}$$

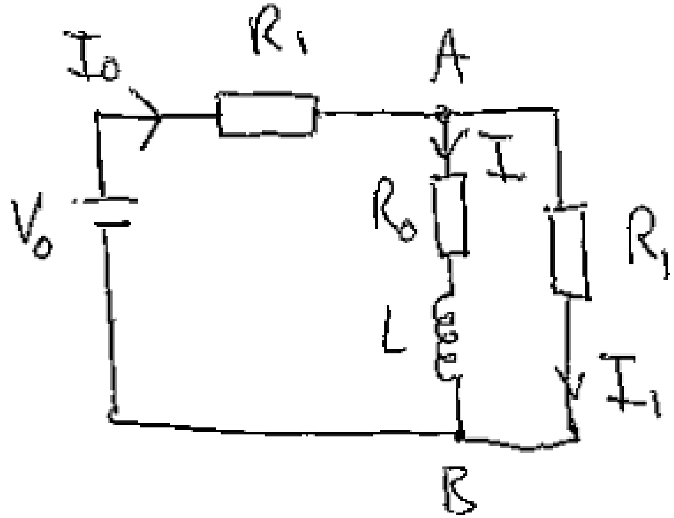
$$I(t) = \frac{V_0}{R}\left(1 - \exp\left(-\frac{t}{\tau}\right)\right)$$



$$I(t) = \frac{V_0}{R}\left(1 - \exp\left(-\frac{t}{\tau}\right)\right)$$

$$\tau = \frac{L}{R}$$





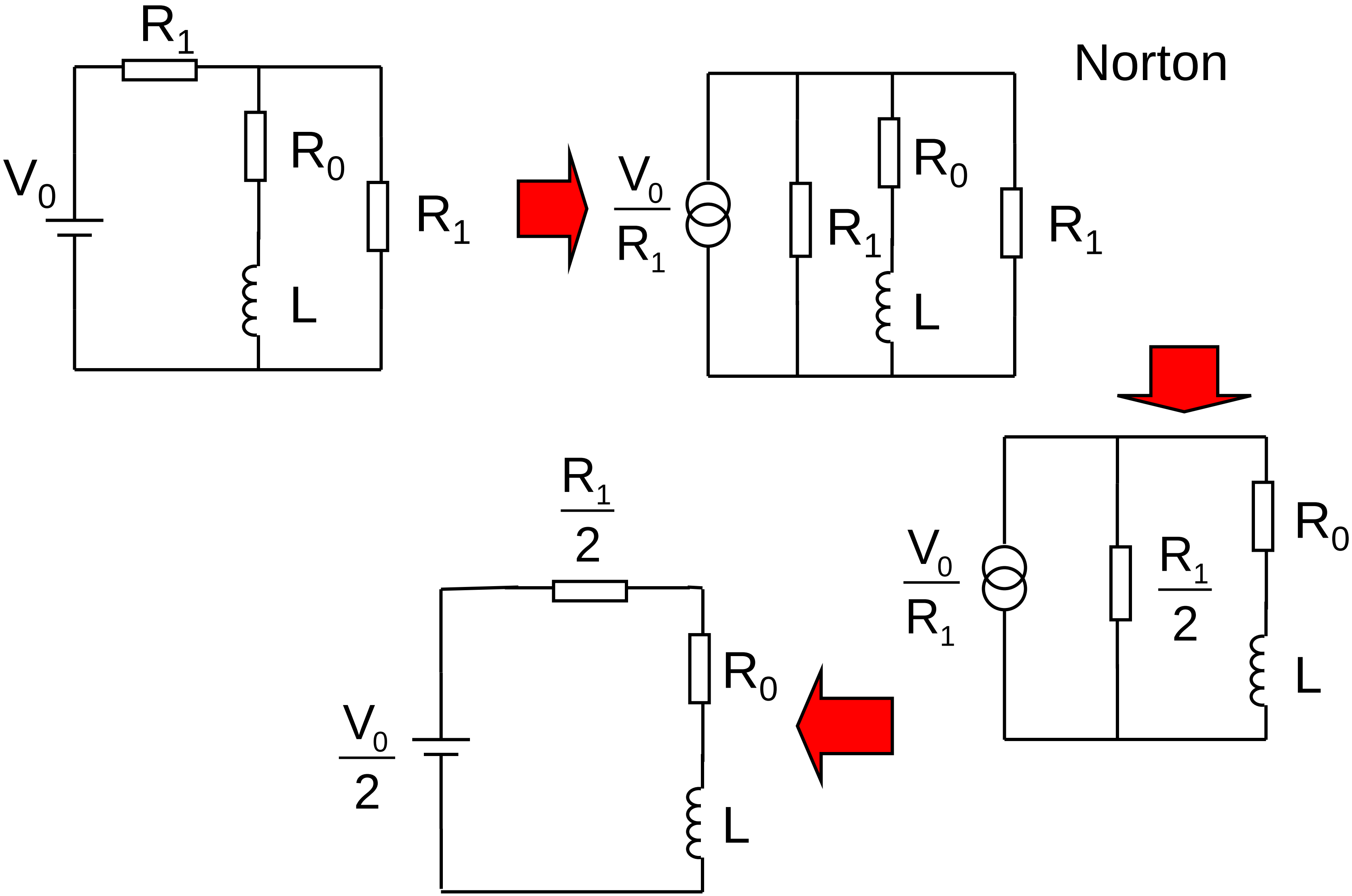
$$I_0 = I + I_1 = \frac{V_0 - V_{AB}}{R_1}$$

$$V_{AB} = I R_0 + L\frac{dI}{dt}$$
$$= I_1 R_1$$

$$\frac{1}{R_1}\left(V_0 - \left(IR_0 + L\frac{dI}{dt}\right)\right) = \frac{1}{R_1}\left(IR_0 + L\frac{dI}{dt}\right) + I$$

$$\frac{V_0}{R_1} = \frac{2}{R_1}\left(IR_0 + L\frac{dI}{dt}\right) + I$$

$$\frac{V_0}{2} = I\left(\frac{R_1}{2} + R_0\right) + L\frac{dI}{dt}$$



$$\frac{V_0}{2} = I\left(\frac{R_1}{2} + R_0\right) + L\frac{dI}{dt}$$

$$\frac{V_0}{2} = I\left(\frac{R_1}{2}+R_0\right) + L\frac{dI}{dt}$$

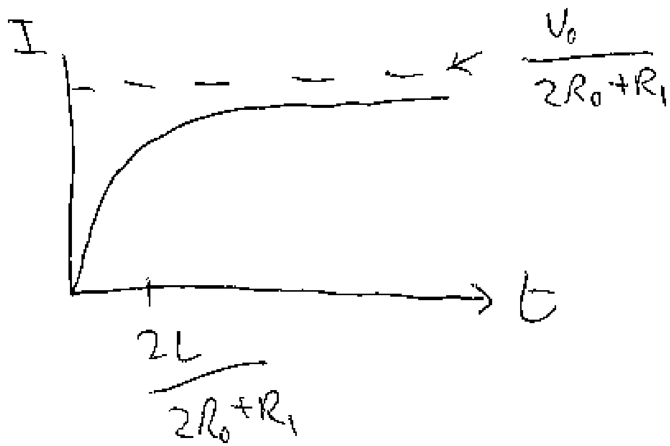
ii) Solve for I

$$\int_0^I \frac{dI}{\frac{V_0}{2} - I\left(\frac{R_1}{2}+R_0\right)} = \frac{1}{L}\int_0^t dt$$

$$-\left[\ln\left(\frac{V_0}{2} - I\left(\frac{R_1}{2}+R_0\right)\right)\right]_0^I \frac{1}{\frac{R_1}{2}+R_0} = \frac{t}{L}$$

$$\left[\frac{V_0}{2} - I\left(\frac{R_1}{2}+R_0\right)\right]\frac{1}{V_0/2} = \exp\left(-\frac{(R_1+2R_0)}{2L}t\right)$$

$$I = \frac{V_0}{2R_0+R_1}\left[1-\exp\left(-\frac{2R_0+R_1}{2L}\right)t\right]$$



iii) V_{AB} at $t = \dfrac{2L}{2R_0 + R_1}$?

$$V_{AB} = I R_0 + L\frac{dI}{dt}$$

$$= \frac{V_0 R_0}{2R_0 + R_1}\left[1 - \exp\left(-\frac{2R_0 + R_1}{2L}t\right)\right] + \frac{V_0}{2}\exp\left[-\frac{2R_0 + R_1}{2L}t\right]$$

$$V_{AB}\left(t = \frac{2L}{2R_0 + R_1}\right) = \frac{V_0 R_0}{2R_0 + R_1}\left(1 - \frac{1}{e}\right) + \frac{V_0}{2e}$$

Energy stored per unit volume

Inductor

$$W = \tfrac{1}{2} L I^2$$

$$= \frac{\mu_0}{2} A \frac{N^2}{\ell} I_0^2$$

$$= \tfrac{1}{2} \mu_0 A \ell H^2 \qquad H = \frac{B}{\mu_0} = \frac{NI}{\ell}$$

$$= \frac{\mu_0}{2} H^2 \cdot \mathrm{Vol}$$

Capacitor

$$W = \tfrac{1}{2} C V^2 \qquad \left[C = \frac{\varepsilon_0 A}{d} \right]$$

$$= \tfrac{1}{2} \frac{\varepsilon_0 A}{d} V^2 \qquad \left[E = \frac{V}{d} \right]$$

$$= \frac{\varepsilon_0}{2} E^2 \cdot \mathrm{Vol}$$