Vectors and Matrices Problems

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Problem Set 2: Vectors Part II

1. Find the cartesian equation for the plane passing through $P_1 = (2, -1, 1), P_2 = (3, 2, -1), P_3 = (-1, 3, 2)$.

2. The position vectors $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \mathbf{b} = -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, and $\mathbf{c} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ all lie in a plane. Find the vector equation of the plane.

3. Identify the following surfaces:
(a) $|\mathbf{r}| = k$
(b) $\mathbf{r} \cdot \mathbf{u} = l$
(c) $\mathbf{r} \cdot \mathbf{u} = m|\mathbf{r}|$ for $-1 \leq m \leq +1$
Here k, l, m are fixed scalars and $\mathbf{u}$ is a fixed unit vector.

4. Find the equation of the line passing through $A = (1, 2, 3)$ perpendicular to the plane $x - 2y + z = 1$.

5. Calculate the shortest distance between the planes:
(1) $x + 2y + 3z = 1$ and (2) $x + 2y + 3z = 5$.

6. A line intersects a plane at an angle $\alpha = 2\pi/3$. The line is defined by $\mathbf{r} = \mu\hat{\mathbf{n}}$ and the plane by $\mathbf{r} \cdot \hat{\mathbf{m}} = 0$, with $\hat{\mathbf{n}}$ and $\hat{\mathbf{m}}$ unit vectors. Calculate the shortest distance from the plane to the point on the line with $\mu = 2$.

7. The plane P_1 contains the points A, B and C, which have position vectors $\mathbf{a} = -3\mathbf{i} + 2\mathbf{j}, \mathbf{b} = 7\mathbf{i} + 2\mathbf{j}$ and $\mathbf{c} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, respectively. Plane P_2 passes through A and is orthogonal to the line BC, whilst plane P_3 passes through B and is orthogonal to the line AC. Find the coordinates of $\mathbf{r}$, the point of intersection of the three planes.

8. Using vector methods show that the line of intersection of the planes $x+2y+3z = 0$ and $3x + 2y + z = 0$ is equally inclined to the $x-$ and $z-$axes and makes an angle $\cos^{-1}(-2/\sqrt{6})$ with the $y-$axis.

9. A line goes from the origin to $P = (1, 1, 1)$; a plane goes through points $A = (-1, 1, -2)$, $B = (1, 5, -5)$, $C = (0, 2, -3)$. Find the intersection point of the plane and the line.

10. Prove that the three vectors $\mathbf{a} = \mathbf{i} + \mathbf{j} + 2\mathbf{k},\ \mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k},\ \mathbf{c} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ are coplanar.

11. For $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$, find a vector which is coplanar with $\mathbf{a}$ and $\mathbf{b}$, but perpendicular to $\mathbf{a}$.

12. The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are not coplanar. The vectors $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$ are the associated *reciprocal vectors*. Verify that the expressions

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}, \quad \mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}, \quad \mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}$$

define a set of reciprocal vectors $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$ with the following properties:

(a) $\mathbf{a}' \cdot \mathbf{a} = \mathbf{b}' \cdot \mathbf{b} = \mathbf{c}' \cdot \mathbf{c} = 1$

(b) $\mathbf{a}' \cdot \mathbf{b} = \mathbf{a}' \cdot \mathbf{c} = \mathbf{b}' \cdot \mathbf{a} = \mathbf{b}' \cdot \mathbf{c} = \mathbf{c}' \cdot \mathbf{a} = \mathbf{c}' \cdot \mathbf{b} = 0$

(c) $\mathbf{a}' \cdot (\mathbf{b}' \times \mathbf{c}') = 1/[\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]$

(d) $\mathbf{a} = (\mathbf{b}' \times \mathbf{c}')/[\mathbf{a}' \cdot (\mathbf{b}' \times \mathbf{c}')]$.

13. Given a set of (non-orthogonal) base vectors $\mathbf{a} = \mathbf{j} + \mathbf{k}, \mathbf{b} = \mathbf{i} + \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j}$:

(a) Establish their reciprocal vectors and hence express the vectors $\mathbf{p} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{q} = \mathbf{i} + 4\mathbf{j}$ and $\mathbf{r} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$ in terms of the base vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(b) Verify that the scalar product $\mathbf{p} \cdot \mathbf{q}$ has the same value, -5, when evaluated using either set of components.

14. Which of the following set of vectors are i) linearly independent and ii) orthogonal (or both) and explain why:

a) $|a_1\rangle = (0, 1, 0), |a_2\rangle = (1, 0, 0)$ and $|a_3\rangle = (0, 0, 1)$

b) $|a_1\rangle = (0, 1, 1), |a_2\rangle = (1, 1, 1)$ and $|a_3\rangle = (0, 0, 1)$

c) $|a_1\rangle = (1, 1, 1), |a_2\rangle = (1, -1, 1)$ and $|a_3\rangle = (1, 1, -2)$

d) $|a_1\rangle = (1, 0, 1), |a_2\rangle = (2, 3, 1)$ and $|a_3\rangle = (1, 6, -1)$.