Vectors and Matrices Problems

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Problem Set 1: Vectors Part I

1. Find the angle between the position vectors to the points $(3, -4, 0)$ and $(-2, 1, 0)$ and find the direction cosines of a vector perpendicular to both.

2. Prove the following by considering vector components:
(a) $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = -(\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c}\,,$
(b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}\,,$
(c) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$ (Lagrange's identity).

3. Prove the following using vector algebra methods:
(a) if $\mathbf{a} \times \mathbf{b} = \mathbf{a} - \mathbf{b}$, this implies that $\mathbf{a} = \mathbf{b}\,,$
(b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}\,,$
(c) if $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$, this implies that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0\,,$
(d) if $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$, this implies that $\mathbf{c} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{b} = \pm|\mathbf{c}| \cdot |\mathbf{a} - \mathbf{b}|\,,$
(e) $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{b}) = \mathbf{b}[\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})]$.

4. Prove using vector algebra methods:

i) The diagonals of a parallelogram bisect each other.

ii) The diagonals of a rhombus are perpendicular to each other.

iii) Two lines, drawn from the end-points of the line of diameter of a circle to a common end-point on the circumference, intersect at a right angle.

5. The vectors **a**, **b**, **c** form the sides of a triangle.

i) Show that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{b} \times \mathbf{c}| = |\mathbf{c} \times \mathbf{a}|$.

ii) Find the area of the triangle having vertices at
$\mathrm{P} = (1, 3, 2)$, $Q = (2, -1, 1)$, $\mathrm{R} = (-1, 2, 3)$.

6. Show that the points $(1, 0, 1), (1, 1, 0)$ and $(1, -3, 4)$ lie on a straight line. Give the equation of the line in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$.

7. What is the shortest distance of $\mathbf{p} = (2, 3, 4)$ from the x-axis?

8. Write down the vector equation of the line

$$\frac{(x-2)}{4} = \frac{(y-1)}{3} = \frac{(z-5)}{2}$$

and find the minimum distance of this line from the origin.

9. Derive an expression for the shortest distance between the two lines $\mathbf{r}_i = \mathbf{q}_i + \lambda_i \mathbf{m}_i, \quad i = 1, 2$. Hence find the shortest distance between the lines

$$\frac{(x-2)}{2} = (y-3) = \frac{(z+1)}{2}$$

and

$$(x+2) = \frac{(y+1)}{2} = (z-1)$$

10. Find the values of the parameters λ and μ at the points of closest approach of the lines $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + \mathbf{j})$ and $\mathbf{r} = \mathbf{j} + 2\mathbf{k} + \mu(-\mathbf{j} + \mathbf{k})$.

11. Two lines are defined by the equations

$$\frac{(x-1)}{2} = (y+2) = \frac{(2-z)}{2}$$

and

$$\frac{(\alpha - x)}{4} = \frac{(y-3)}{\beta} = \frac{(z+4)}{4},$$

where α and β are undetermined constants.

a) Why isn't an intersection of the lines possible if $\beta = -2$?

b) If $\beta = 1$, what condition must α satisfy for the lines to intersect?

12. Two small objects travel with equal speed. The first starts from the point $(2, 3, 3)$ and travels in the direction $(-1, -1, 0)$, while the second starts at the same time from $(3, 2, 1)$ and travels in the direction $(-1, 0, 1)$. Determine whether or not they collide.