CP3 REVISION LECTURES

VECTORS AND MATRICES

Lecture 2

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OUTLINE

5. *Solutions to simultaneous linear equations*

6. *Rotation and matrix operators*

7. *Eigenvalues and Eigenvectors*

8. *Diagonalization of a matrix*

5. Solutions to simultaneous linear equations

- We can write the set of simultaneous linear equations as a matrix equation:
  $A\mathrm{x} = b$, (A is called the *coefficient matrix*). i.e.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{pmatrix} \qquad (1)$$

  where a_{ij} and b_i have known values, x_i are unknown.
- We can define the *augmented matrix*

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{1n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right) \qquad (2)$$

- If the b_i are all zero, then the system of equations is called *homogeneous*, otherwise its *inhomogeneous*.

Unique solutions to simultaneous equations

- Consider $N = 3$

$$\begin{array}{ccccccc} a_{11}x & + & a_{12}y & + & a_{13}z & = & b_1 \\ a_{21}x & + & a_{22}y & + & a_{23}z & = & b_2 \\ a_{31}x & + & a_{32}y & + & a_{33}z & = & b_3 \end{array} \tag{3}$$

- Condition for the solution to be unique:
  - [Rank of coefficient matrix] = [Rank of augmented matrix] = = [Number of unknowns]
  - OR alternatively $|A| \neq 0$ and $\underline{\mathbf{b}} \neq 0$.
- Note that $|A| \neq 0$ and $\underline{\mathbf{b}} = \mathbf{0}$ gives the trivial solution $(x, y, z) = (0, 0, 0)$.

Unique solution: matrix inversion method

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{pmatrix} \qquad (4)$$

- The equations are written $A\mathrm{x} = b$, therefore we write $\mathrm{x} = A^{-1}b$ where $(A^{-1})_{ij} = (C^T)_{ij}/|A|$ as before.
- Hence evaluate A^{-1} and the solutions drop out trivially
- Note the following:
  - The method needs $|A|$ to be $\neq 0$ (i.e. non-singular),
  - If all the $b_i = 0$, only the trivial solution $x_i = 0$ will be found.

Unique solution : Cramer's method

$$\begin{array}{ccccccc} a_1 x & + & b_1 y & + & c_1 z & = & v_1 \\ a_2 x & + & b_2 y & + & c_2 z & = & v_2 \\ a_3 x & + & b_3 y & + & c_3 z & = & v_3 \end{array} \Rightarrow \mathbf{A}\underline{\mathbf{x}} = \underline{\mathbf{v}} \qquad (5)$$

Define Cramer's determinant $\rightarrow$ $|A|$ with columns replaced by the RHS of equations:

$$\Delta_x = \begin{vmatrix} v_1 & b_1 & c_1 \\ v_2 & b_2 & c_2 \\ v_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_y = \begin{vmatrix} a_1 & v_1 & c_1 \\ a_2 & v_2 & c_2 \\ a_3 & v_3 & c_3 \end{vmatrix}, \quad \Delta_z = \begin{vmatrix} a_1 & b_1 & v_1 \\ a_2 & b_2 & v_2 \\ a_3 & b_3 & v_3 \end{vmatrix} \qquad (6)$$

$$\text{and} \quad |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad (7)$$

Solution is then:

$x = \Delta_x/|A|, \quad y = \Delta_y/|A|, \quad z = \Delta_z/|A|.$

Solutions do not exist

- Solutions do not exist if:
  - $|A| = 0$ and $\underline{\mathbf{b}} \neq 0$ and
    [Rank of coefficient matrix] $<$ [Rank of augmented matrix]
  - i.e. $|A| = 0$ and any of Cramer's determinants are not equal to zero (*)

    (*) since it is the Cramer's determinants (either $= 0$ or $\neq 0$) which determine the rank of the augmented matrix.

Example: CP3 September 2007. No. 8

For which value of c does the set of linear equations

$$\begin{aligned} 2x + y - 2z &= 1 \\ -2x + 3y + z &= 3 \\ cx + 4y - z &= d \end{aligned}$$

not have a unique solution? Give a geometrical interpretation of the set of equations for this value of c distinguishing the cases $d = 4$ and $d \neq 4$. *[8]*

- No unique solution if $\begin{vmatrix} 2 & 1 & -2 \\ -2 & 3 & 1 \\ c & 4 & -1 \end{vmatrix} = 0$
- Hence $(2\times-7)-1\times(2-c)+(-2)\times(-8-3c) = -14-2+c+16+6c = 0$
- No unique solution for $c = 0$

CP3 September 2007. No. 8, continued

- $d = 4, \quad |A| = 0$

$$\begin{array}{rcrcrcl} 2x & + & y & - & 2z & = & 1 \\ -2x & + & 3y & + & z & = & 3 \\ & & 4y & - & z & = & 4 \end{array} \quad \rightarrow \quad \left(\begin{array}{ccc|c} 2 & 1 & -2 & 1 \\ -2 & 3 & 1 & 3 \\ 0 & 4 & -1 & 4 \end{array} \right)$$

- Rank of coefficient matrix = 2
- Get rank of augmented matrix

Cramer's determinants, Δ_z, Δ_x: $\begin{vmatrix} 2 & 1 & 1 \\ -2 & 3 & 3 \\ 0 & 4 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -2 \\ 3 & 3 & 1 \\ 4 & 4 & -1 \end{vmatrix} = 0$

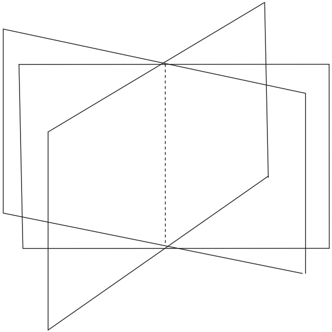
(since two columns are equal).

And Δ_y: $\begin{vmatrix} 2 & 1 & -2 \\ -2 & 3 & 1 \\ 0 & 4 & -1 \end{vmatrix} = 0$ (since it's identical to $|A|$)

- Hence rank of augmented matrix = 2

CP3 September 2007. No. 8, continued

- $d = 4$: All three planes meet on a common line



- Since all Cramer's determinants are zero, AND no single equation is a multiple of the other.
- An infinite number of solutions.

CP3 September 2007. No. 8, continued

- $d \neq 4, \quad |A| = 0$

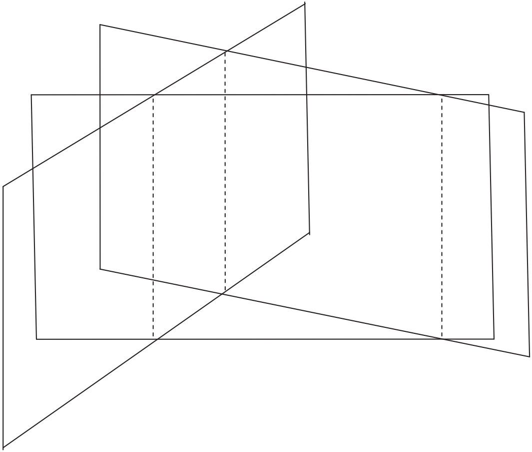
$$\begin{array}{rcrcrcl} 2x & + & y & - & 2z & = & 1 \\ -2x & + & 3y & + & z & = & 3 \\ & & 4y & - & z & = & d \end{array} \quad \rightarrow \quad \left(\begin{array}{ccc|c} 2 & 1 & -2 & 1 \\ -2 & 3 & 1 & 3 \\ 0 & 4 & -1 & d \end{array} \right)$$

- Rank of coefficient matrix = 2
- Get rank of augmented matrix

Cramer's determinants: e.g $\Delta_z = \begin{vmatrix} 2 & 1 & 1 \\ -2 & 3 & 3 \\ 0 & 4 & d \end{vmatrix} \neq 0$

- Hence, [Rank of coefficient matrix] $<$ [Rank of augmented matrix]

CP3 September 2007. No. 8, continued



- Lines of intersection of the planes are parallel to each other.
- No solutions exist

Homogeneous equations

- $|A| = 0$ and $\underline{\mathbf{b}} = 0$

$$\begin{array}{ccccccc} a_{11}x & + & a_{12}y & + & a_{13}z & = & 0 \\ a_{21}x & + & a_{22}y & + & a_{23}z & = & 0 \\ a_{31}x & + & a_{32}y & + & a_{33}z & = & 0 \end{array} \quad (8)$$

- $\underline{\mathbf{b}} = 0$ gives the trivial solution $(x, y, z) = (0, 0, 0)$ *unless* $|A| = 0$
- Three planes meet on a common line passing through the origin, note that only the *ratios* $x/y,\ x/z,\ y/z$ can be found.
- Example

$$\begin{array}{cccccccc} 2x & + & 3y & + & 4z & = & 0 & (1) \\ x & + & 2y & + & 2z & = & 0 & (2) \\ -x & + & y & - & 2z & = & 0 & (3) \end{array} \quad (9)$$

$|A| = 0$ and $\underline{\mathbf{b}} = 0$
Line through the origin is $y = 0,\ x = -2z$

6. Rotation and matrix operators

- We can write a transformation in matrix form:

$$x = Sx'$$

where S is a *transformation matrix*.
This transforms the change of basis, and also transforms the vector components $x' \rightarrow x$.

- The inverse transformation transforms x back to x', leaving it unchanged by the two successive transformations.

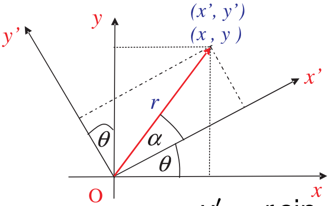
$$x' = S^{-1}x$$

Example: CP3 September 2009. No. 10

First part: *The axes of a coordinate system* (x', y') *are rotated by an angle* θ *in the counter-clockwise direction with respect to the axes of a coordinate system* (x, y)*, and the two systems share a common origin. Show that the coordinates* x' *and* y' *can be expressed in terms of* x *and* y *using the relation*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{R}(\theta) \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{where} \quad \mathbf{R}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Show that $\mathbf{R}^{-1} = \mathbf{R}^T$*, where* $\mathbf{R}^T$ *is the transpose of* $\mathbf{R}$*.* [5]



$$x' = r\cos\alpha$$
$$x = r\cos(\theta+\alpha)$$
$$\rightarrow \quad x' = \frac{x\cos\alpha}{\cos(\theta+\alpha)}$$
$$x\cos\alpha = x'\cos\theta\cos\alpha - x'\sin\theta\sin\alpha$$
Since $x'\sin\alpha = y'\cos\alpha$

$$x = x'\cos\theta - y'\sin\theta$$

$$y' = r\sin\alpha$$
$$y = r\sin(\theta+\alpha)$$
$$\rightarrow \quad y' = \frac{y\sin\alpha}{\sin(\theta+\alpha)}$$
$$y\sin\alpha = y'\sin\theta\cos\alpha + y'\cos\theta\sin\alpha$$
Since $y'\cos\alpha = x'\sin\alpha$

$$y = x'\sin\theta + y'\cos\theta$$

- Coordinate transformation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \quad (10)$$

- Take the inverse:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (11)$$

These equations relate the coordinates of $\underline{\mathbf{r}}$ measured in the (x, y) frame with those measured in the rotated (x', y') frame

Rotation of a vector in fixed 3D coord. system

- In 3D, we can rotate a vector $\underline{\mathbf{r}}$ about any one of the three axes

$$\underline{\mathbf{r}}' = R(\theta)\,\underline{\mathbf{r}}$$

A rotation about the z axis is given by

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{12}$$

- For rotations about the x and y axes

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \quad R_y(\gamma) = \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix} \tag{13}$$

- But note that now for successive rotations:

$$R_z(\theta)R_x(\alpha) \neq R_x(\alpha)R_z(\theta)$$

Matrices and quadratic forms

Example: CP3 September 2009. No. 10

Second part: *The equation of an ellipse whose major axis is inclined at an angle with the respect to the x-axis may be written as* $f(x, y) = 2x^2 + 2y^2 - 2xy = 9$. *Find the elements of the symmetric matrix* M *that satisfies the relation* $(x, y)\,\mathrm{M}\begin{pmatrix} x \\ y \end{pmatrix}$. *[3]*

- Write $X^T AX$ in generalized form:

$$(x, y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (x, y)\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = ax^2 + bxy + cxy + dy^2$$

- Compare coefficients $\rightarrow a = 2, d = 2, (b + c) = -2$
  Write in symmetrical form $b = c = -1$
- Hence in matrix representation:
  $$(x, y)\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 9$$

7. Eigenvalues and Eigenvectors

- An eigenvalue equation is one that transforms as

$$A|\mathbf{x}\rangle = \lambda|\mathbf{x}\rangle$$

  where λ is just a number (can be complex)

  - A has transformed $|\mathbf{x}\rangle$ into a multiple of itself
  - Vector $|\mathbf{x}\rangle$ is the *eigenvector* of the operator A
    λ is the *eigenvalue*.
  - The operator A can have in principle a *series* of eigenvectors $|\mathbf{x_j}\rangle$ and eigenvalues λ_j.

- Write in matrix form:

$$Ax = \lambda x$$

  where A is an $N \times N$ matrix.

- In QM, often deal with *normalized* eigenvectors:
  $x^\dagger x = \langle \mathbf{x}|\mathbf{x}\rangle = 1$ (where $x^\dagger = x^{*T} \rightarrow$ Hermitian conjugate)

Finding eigenvalues and eigenvectors

- Eigenvalue equation:

$$Ax = \lambda x \;=\; \lambda I x \quad (I \text{ is the unit matrix})$$

  - $Ax - \lambda I x = 0$
  - $(A - \lambda I)x = 0$
  - A set of linear simultaneous equations of degree N.

- Homogeneous equations only have a non-trivial solution (x_i non-zero) if the determinant

$$|A - \lambda I| = 0$$

Example: CP3 June 2010. No. 4

Find the eigenvalues and normalized eigenvectors of the two-dimensional rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ *where* θ *is the (real) rotation angle. Show explicitly that the eigenvectors are orthogonal.* *[8]*

- First the eigenvalues: start from $|A - \lambda I| = 0$

$$\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$$

- $(\cos\theta - \lambda)^2 + \sin^2\theta = \lambda^2 - 2\lambda\cos\theta + 1 = 0$
- $\lambda = \cos\theta \pm \frac{1}{2}\sqrt{4\cos^2\theta - 4} = \cos\theta \pm i\sin\theta$
- $\lambda = e^{\pm i\theta}$

CP3 June 2010. No. 4, continued

- Now find the eigenvectors - substitute into the eigenvalue equation:
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (\cos\theta \pm i\sin\theta) \begin{pmatrix} x \\ y \end{pmatrix}$$
- $x\cos\theta - y\sin\theta = x(\cos\theta \pm i\sin\theta)$
$x\sin\theta + y\cos\theta = y(\cos\theta \pm i\sin\theta)$
- $-y\sin\theta = \pm ix\sin\theta$
$x\sin\theta = \pm iy\sin\theta$
- $x/y = \pm i$
- Set $x = 1 \quad \rightarrow \quad$ Eigenvectors :
$$\psi_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ +i \end{pmatrix}, \quad \psi_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$
(Eigenvectors should be multiplied by some arbitrary phase $e^{i\alpha}$)
(Normalization $\frac{1}{\sqrt{2}}$ comes from conditions $\psi_+^\dagger \psi_+ = 1,\ \psi_-^\dagger \psi_- = 1$)

CP3 June 2010. No. 4, continued

- Orthgonality of Eigenvectors
- From before : $\psi_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ +i \end{pmatrix}, \ \ \psi_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$
- $\psi_+^\dagger \psi_- = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \left(\ (1, -i) \ \right) \begin{pmatrix} 1 \\ -i \end{pmatrix} = 0$

Eigenvalues and eigenvectors of an Hermitian matrix

- Hermitian conjugate of a matrix: $A^\dagger = (A^T)^* = (A^*)^T$
  A complex matrix with $A = A^\dagger$ is *Hermitian*.

- The eigenvalues of Hermitian matrix are real

- The eigenvectors of Hermitian matrix are orthogonal

(See lecture notes for proofs)

8. Diagonalization of a matrix

To “diagonalize” a matrix:

- Take a given $N \times N$ matrix A
- Construct a $N \times N$ matrix S that has the eigenvectors of A as its columns
- Then the "similarity transformation" matrix $A' \rightarrow (S^{-1}AS)$ is diagonal and has the *eigenvalues* of A as its diagonal elements.

Example: adapted from CP3 June 2010. No. 10

Let the columns of the matrix S be the normalized eigenvectors of the Hermitian matrix A. Show that $D = S^{-1}AS$ is a diagonal matrix. What are the diagonal elements of D? *[5]*

- x_j, λ_j are the eigenvectors/values of operator A: $Ax_j = \lambda_j x_j$
- Consider a *similarity transformation* from some basis $|\mathbf{e}\rangle \to |\mathbf{e}'\rangle$ $A \to A' = S^{-1}AS$, where the columns j of the matrix S are the special case of the *eigenvectors* of the matrix A, $\begin{pmatrix} \uparrow & \uparrow & \cdots \\ x_1 & x_2 & \cdots \\ \downarrow & \downarrow & \cdots \end{pmatrix}$ i.e. $S_{ij} \equiv (x_j)_i$ (for the i^{th} component of x_j).
- Consider the individual elements of $S^{-1}AS$ in this case

$$\begin{aligned} A'_{ij} &= (S^{-1}AS)_{ij} \\ &= \textstyle\sum_k (S^{-1})_{ik} (\sum_m A_{km} S_{mj}) = \sum_k \sum_m (S^{-1})_{ik} A_{km} S_{mj} \\ &= \textstyle\sum_k \sum_m (S^{-1})_{ik} A_{km} (x_j)_m = \sum_k (S^{-1})_{ik} \lambda_j (x_j)_k \\ &= \textstyle\sum_k \lambda_j (S^{-1})_{ik} S_{kj} = \lambda_j \delta_{ij} \end{aligned}$$

where δ_{ij} is the Kronecker delta.

Hence $S^{-1}AS$ is a diagonal matrix with the eigenvalues of A along the diagonal.