CP3 REVISION LECTURES

VECTORS AND MATRICES

Lecture 1

Prof. N. Harnew

University of Oxford

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OUTLINE

1. *Vector Algebra*

2. *Vector Geometry*

3. *Types of Matrices and Matrix Operations*

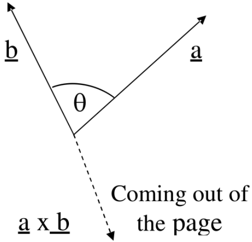
4. *Determinants and matrix inverses*

1. Vector Algebra

Scalar (or dot) product definition:
$\underline{\mathbf{a}}.\underline{\mathbf{b}} = |\underline{\mathbf{a}}|.|\underline{\mathbf{b}}| \cos\theta \equiv ab\cos\theta$

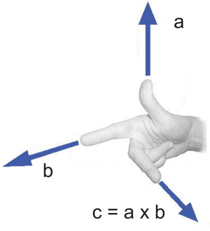
- $\underline{\mathbf{a}}.\underline{\mathbf{b}} = a_x b_x + a_y b_y + a_z b_z$

Vector (or cross) product definition:
$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = |\underline{\mathbf{a}}||\underline{\mathbf{b}}| \sin\theta\, \underline{\hat{\mathbf{n}}}$



- To get direction of $\underline{\mathbf{a}} \times \underline{\mathbf{b}}$ use right hand rule
- $\underline{\hat{\mathbf{n}}}$ is a *unit vector* in a direction *perpendicular* to both $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$

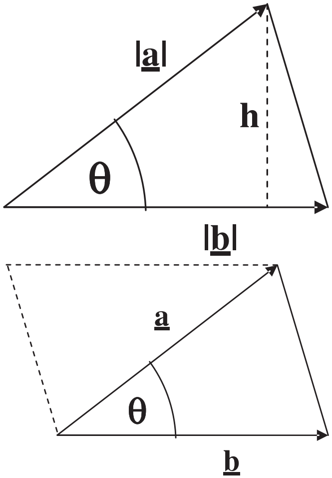
$$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$



Geometrical interpretation of vector product

Vector product is related to the area of a triangle:

- Height of triangle $h = a\sin\theta$
- Area of triangle = A_{triangle} = 1/2 $\times$ base $\times$ height $= \frac{bh}{2} = \frac{ab\,\sin\theta}{2} = \frac{|\underline{\mathbf{a}}\times\underline{\mathbf{b}}|}{2}$
- Vector product therefore gives the area of the parallelogram: $A_{\text{parallelogram}} = |\underline{\mathbf{a}} \times \underline{\mathbf{b}}|$



Scalar and vector triple products

The scalar triple product $\underline{\mathbf{a}}.(\underline{\mathbf{b}} \times \underline{\mathbf{c}}) \quad (\equiv [\underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}}])$

- In determinant form: $\underline{\mathbf{a}}.(\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$
- *Cyclic* permutations of $\underline{\mathbf{a}}$, $\underline{\mathbf{b}}$ and $\underline{\mathbf{c}}$ leaves the triple scalar product unaltered: $\underline{\mathbf{a}}.(\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = \underline{\mathbf{c}}.(\underline{\mathbf{a}} \times \underline{\mathbf{b}}) = \underline{\mathbf{b}}.(\underline{\mathbf{c}} \times \underline{\mathbf{a}})$
  Non-cyclic permutations change the sign of the STP

The vector triple product $\underline{\mathbf{a}} \times (\underline{\mathbf{b}} \times \underline{\mathbf{c}})$

- This is *not* associative. i.e. $\underline{\mathbf{a}} \times (\underline{\mathbf{b}} \times \underline{\mathbf{c}}) \neq (\underline{\mathbf{a}} \times \underline{\mathbf{b}}) \times \underline{\mathbf{c}}$
- It can be shown using components:

$$\underline{\mathbf{a}} \times (\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = (\underline{\mathbf{a}}.\underline{\mathbf{c}})\, \underline{\mathbf{b}} - (\underline{\mathbf{a}}.\underline{\mathbf{b}})\, \underline{\mathbf{c}}$$

This identity is given on the Prelims data sheet.

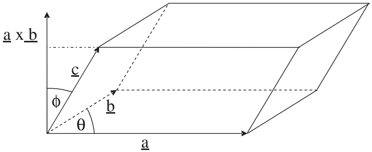
Example: CP3 June 2010. No. 4

Show that for any four vectors $\underline{\mathbf{a}}$, $\underline{\mathbf{b}}$, $\underline{\mathbf{c}}$, and $\underline{\mathbf{d}}$,
$(\underline{\mathbf{a}} \times \underline{\mathbf{b}}) \cdot (\underline{\mathbf{c}} \times \underline{\mathbf{d}}) = (\underline{\mathbf{a}} \cdot \underline{\mathbf{c}})(\underline{\mathbf{b}} \cdot \underline{\mathbf{d}}) - (\underline{\mathbf{a}} \cdot \underline{\mathbf{d}})(\underline{\mathbf{b}} \cdot \underline{\mathbf{c}})$. [4]

- $(\underline{\mathbf{a}} \times \underline{\mathbf{b}}) \cdot (\underline{\mathbf{c}} \times \underline{\mathbf{d}}) = \underline{\mathbf{d}} \cdot ((\underline{\mathbf{a}} \times \underline{\mathbf{b}}) \times \underline{\mathbf{c}}) = \underline{\mathbf{c}} \cdot (\underline{\mathbf{d}} \times (\underline{\mathbf{a}} \times \underline{\mathbf{b}}))$
  (Using properties of scalar triple product)
- $= \underline{\mathbf{c}} \cdot ((\underline{\mathbf{d}} \cdot \underline{\mathbf{b}})\, \underline{\mathbf{a}} - (\underline{\mathbf{d}} \cdot \underline{\mathbf{a}})\, \underline{\mathbf{b}})$
  (Using identity of vector product)
- $= (\underline{\mathbf{a}} \cdot \underline{\mathbf{c}})(\underline{\mathbf{b}} \cdot \underline{\mathbf{d}}) - (\underline{\mathbf{a}} \cdot \underline{\mathbf{d}})(\underline{\mathbf{b}} \cdot \underline{\mathbf{c}})$
  (Rearranging)

Geometrical interpretation of STP

The triple scalar product can be interpreted as the volume of a parallelepiped:



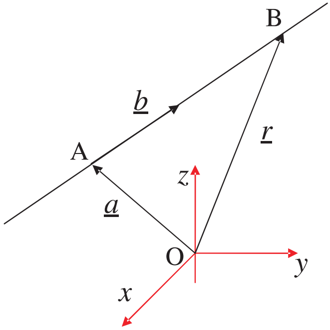
- [Volume] = [Area of base] $\times$ [Vertical height of parallelepiped]
- [Area of base] = $|\underline{\mathbf{a}} \times \underline{\mathbf{b}}|$ (vector direction is perpendicular to the base)
- [Vertical height] $= |\underline{\mathbf{c}}| \cos\phi = \underline{\mathbf{c}} \cdot (\frac{\underline{\mathbf{a}} \times \underline{\mathbf{b}}}{|\underline{\mathbf{a}} \times \underline{\mathbf{b}}|})$
  - Hence [Volume] $= |\underline{\mathbf{a}} \times \underline{\mathbf{b}}| \left(\underline{\mathbf{c}} \cdot (\frac{\underline{\mathbf{a}} \times \underline{\mathbf{b}}}{|\underline{\mathbf{a}} \times \underline{\mathbf{b}}|})\right) = \underline{\mathbf{c}} \cdot (\underline{\mathbf{a}} \times \underline{\mathbf{b}})$
  - Obviously if $\underline{\mathbf{a}}, \underline{\mathbf{b}}$ and $\underline{\mathbf{c}}$ are coplanar, volume = 0.

Representation of lines in vector form

- Point A is any fixed position on the line with position vector $\underline{\mathbf{a}}$. Line direction is defined by vector $\underline{\mathbf{b}}$. Position vector $\underline{\mathbf{r}}$ is a general point on the line.
- The equation of the line is then:

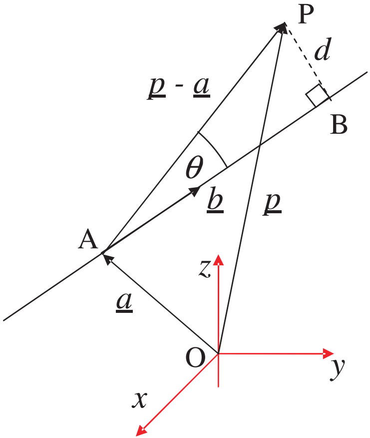
$$\underline{\mathbf{r}} = \underline{\mathbf{a}} + \lambda \underline{\mathbf{b}}$$

where λ takes all values to give all positions on the line.



Distance from a point to a line

- Line is given by $\underline{\mathbf{r}} = \underline{\mathbf{a}} + \lambda\underline{\mathbf{b}}$
- The minimum distance, *d*, from *P* to the line is when angle *ABP* is a right angle.
- From geometry: $d = |\underline{\mathbf{p}} - \underline{\mathbf{a}}| \sin\theta$
- *d* is therefore the magnitude of the vector product $(\underline{\mathbf{p}} - \underline{\mathbf{a}}) \times \underline{\mathbf{b}}/|\underline{\mathbf{b}}|$
- Hence $d = |(\underline{\mathbf{p}} - \underline{\mathbf{a}}) \times \hat{\underline{\mathbf{b}}}|$



Example: CP3 June 2005. No. 6

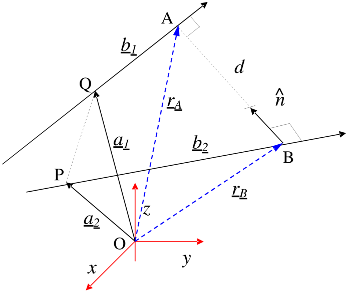
What is the shortest distance from the point $\underline{\mathbf{P}}$ *to the line* $\underline{\mathbf{r}} = \underline{\mathbf{a}} + \lambda \underline{\mathbf{b}}$*? Determine this shortest distance for the case where* $\underline{\mathbf{p}} = (2, 3, 4)$ *and the line is the x-axis.* *[5]*

$\rightarrow$ $d = |(\underline{\mathbf{p}} - \underline{\mathbf{a}}) \times \hat{\underline{\mathbf{b}}}|$ from before.

$\rightarrow$ In the example, line is the x-axis :
$\hat{\underline{\mathbf{b}}} = (1, 0, 0)$; $\underline{\mathbf{a}} = (0, 0, 0)$

$$\rightarrow \mathbf{d} = |((\mathbf{2}, \mathbf{3}, \mathbf{4}) - (\mathbf{0}, \mathbf{0}, \mathbf{0})) \times (\mathbf{1}, \mathbf{0}, \mathbf{0})|$$
$$= \sqrt{(\mathbf{3^2} + \mathbf{4^2})} = \mathbf{5}$$

Minimum distance from a line to a line

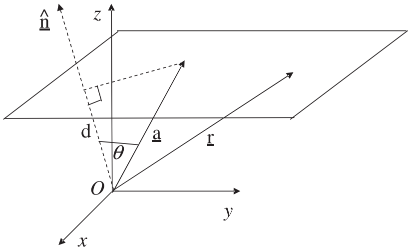


- Two lines in 3D
  $\underline{\mathbf{r_1}} = \underline{\mathbf{a_1}} + \lambda_1 \underline{\mathbf{b_1}}, \quad \underline{\mathbf{r_2}} = \underline{\mathbf{a_2}} + \lambda_2 \underline{\mathbf{b_2}}$
- The shortest distance is represented by the vector perpendicular to both lines
- The unit vector normal to both lines is: $\underline{\hat{\mathbf{n}}} = \frac{\underline{\mathbf{b_1}} \times \underline{\mathbf{b_2}}}{|\underline{\mathbf{b_1}} \times \underline{\mathbf{b_2}}|}$

$$|\underline{\mathbf{d}}| = (\underline{\mathbf{a_1}} - \underline{\mathbf{a_2}}).\underline{\hat{\mathbf{n}}} = (\underline{\mathbf{a_1}} - \underline{\mathbf{a_2}}).\frac{\underline{\mathbf{b_1}} \times \underline{\mathbf{b_2}}}{|\underline{\mathbf{b_1}} \times \underline{\mathbf{b_2}}|}$$

Representation of planes in vector form

- Vector $\underline{\mathbf{a}}$ is any position vector to the plane. $\underline{\mathbf{r}}$ is a position vector to a general point on the plane.



- The vector equation for the plane is written:

$$(\underline{\mathbf{r}} - \underline{\mathbf{a}}).\underline{\hat{\mathbf{n}}} = 0$$

where $\underline{\hat{\mathbf{n}}}$ is the unit vector perpendicular to the plane.

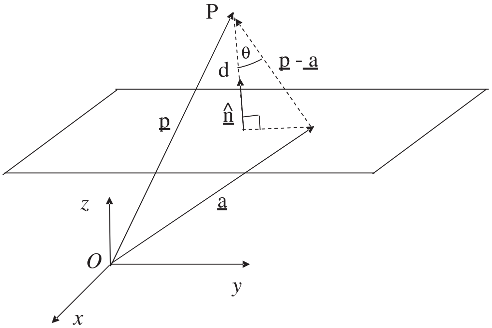
- The plane can also be written as

$$\underline{\mathbf{r}}.\underline{\hat{\mathbf{n}}} = lx + my + nz = d$$

where $\underline{\hat{\mathbf{n}}} = (l, m, n)$, $\underline{\mathbf{r}} = (x, y, z)$ & d is perpendicular distance

Minimum distance from a point to a plane

- Consider vector $(\underline{\mathbf{p}} - \underline{\mathbf{a}})$ which is a vector from the plane $(\underline{\mathbf{r}} - \underline{\mathbf{a}}).\hat{\underline{\mathbf{n}}} = 0$ to the point P

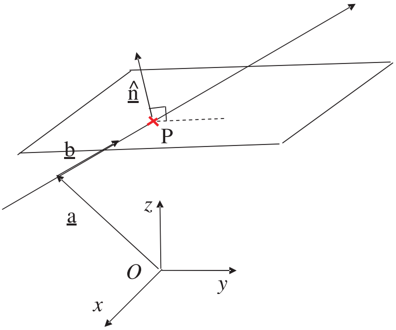


- The component of $(\underline{\mathbf{p}} - \underline{\mathbf{a}})$ normal to the plane is equal to the minimum distance of P to the plane.

i.e. $d = (\underline{\mathbf{p}} - \underline{\mathbf{a}}) \cdot \hat{\underline{\mathbf{n}}}$

(sign depends on which side of plane the point is situated).

Intersection of a line with a plane



- A line is given by $\underline{\mathbf{r}} = \underline{\mathbf{a}} + \lambda\underline{\mathbf{b}}$
- A normal vector to the plane is $\underline{\mathbf{n}} = l\underline{\mathbf{i}} + m\underline{\mathbf{j}} + n\underline{\mathbf{k}}$
- To get the intersection point, substitute equation of line $\underline{\mathbf{r}} = \underline{\mathbf{a}} + \lambda\underline{\mathbf{b}} = (x, y, z) = (a_x + \lambda b_x, a_y + \lambda b_y, a_z + \lambda b_z)$ into equation of plane $lx + my + nz = d$
- Solve for λ and substitute into the equation of the line. This gives the point of intersection.

Example: CP3 June 2008. No. 7

First part: *A line is given by the equation* $\underline{\mathbf{r}} = 3\underline{\mathbf{i}} - \underline{\mathbf{j}} + (2\underline{\mathbf{i}} + \underline{\mathbf{j}} - 2\underline{\mathbf{k}})\lambda$ *where* λ *is a variable parameter and* $\underline{\mathbf{i}}$, $\underline{\mathbf{j}}$, $\underline{\mathbf{k}}$ *are unit vectors along the cartesian axes* x, y, z. *The equation of the plane containing this line and the point (2,1,0) may be expressed in the form* $\underline{\mathbf{r}} \cdot \underline{\hat{\mathbf{n}}} = d$ *where* $\underline{\hat{\mathbf{n}}}$ *is a unit vector and* d *is a constant. Find* $\underline{\hat{\mathbf{n}}}$ *and* d*, and explain their geometrical meaning.* *[5]*

- Two points in the plane are (3,-1,0) (for $\lambda = 0$) and $(5, 0, -2)$ (for $\lambda = 1$)
- Two lines in the plane are
  $(5, 0, -2) - (3, -1, 0) = (2, 1, -2)$ and $(2, 1, 0) - (5, 0, -2) = (-3, 1, 2)$
- Therefore a normal to the plane is: $\begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ 2 & 1 & -2 \\ -3 & 1 & 2 \end{vmatrix} = (4, 2, 5)$

  $\rightarrow$ $\underline{\hat{\mathbf{n}}}$ is $(4, 2, 5)/\sqrt{45}$
- Plane $\underline{\mathbf{r}} \cdot \underline{\hat{\mathbf{n}}} = d$ where
  $d = (3, -1, 0) \cdot ((4, 2, 5)/\sqrt{45}) = (12 - 2)/\sqrt{45} = 10/(3\sqrt{5})$

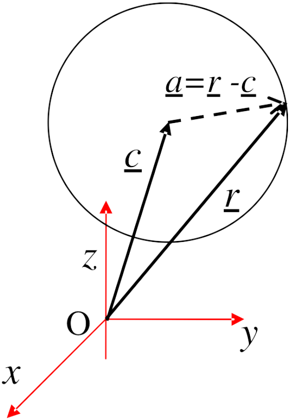
  is the closest distance of plane to origin.

CP3 June 2008. No. 7, continued

Second part: Find the volume of the tetrahedron with its four corners at: the origin, the point (2,1,0), and the points on the line with $\lambda = 0$ and $\lambda = 1$. [5]

- A tetrahedron is a volume composed of four triangular faces, three of which meet at each vertex
- The volume of a tetrahedron is equal to 1/6 of the volume of a parallelepiped that shares three converging edges with it.
- Volume is 1/6 the triple scalar product of $(2, 1, 0), (3, -1, 0)$ and $(5, 0, -2)$
- Volume = $\frac{1}{6}\begin{vmatrix} 2 & 1 & 0 \\ 3 & -1 & 0 \\ 5 & 0 & -2 \end{vmatrix} = (2 \times 2 - 1 \times (-6) + 0)/6 = \frac{5}{3}$

Vector representation of a sphere



$$|\underline{\mathbf{r}} - \underline{\mathbf{c}}|^2 = a^2$$

alternatively

$$r^2 - 2\underline{\mathbf{r}} \cdot \underline{\mathbf{c}} + c^2 = a^2$$

- $\underline{\mathbf{c}}$ is the position vector to the centre of the sphere
- $a = |\underline{\mathbf{a}}|$ is the sphere radius (scalar)

- The two points that are the intersection of the sphere with a line $\underline{\mathbf{r}} = \underline{\mathbf{p}} + \lambda\underline{\mathbf{q}}$ are given by solving the quadratic for λ :

$$(\underline{\mathbf{p}} + \lambda\underline{\mathbf{q}} - \underline{\mathbf{c}}) \cdot (\underline{\mathbf{p}} + \lambda\underline{\mathbf{q}} - \underline{\mathbf{c}}) = a^2$$

3. Types of Matrices and Matrix Operations

a) The diagonal matrix

- A is diagonal if $A_{ij} = 0$ for $i \neq j$ (for a square matrix).
  i.e. the matrix has only elements on the diagonal which are different from zero.

b) The unit matrix

- A diagonal matrix I with all diagonal elements = 1.

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{1}$$

This has the property $AI = IA = A$.

c) Transpose of a matrix

- The transpose of a matrix A is a matrix B with the rows and columns of A interchanged.
- $B = A^T \quad \Rightarrow \quad B_{ji} = A_{ij}$
- $(AB)^T = B^T A^T$ (note that the order of A and B is reversed).

d) Hermitian conjugate

- Take the complex conjugate of the transpose:
  $A^\dagger = (A^T)^* = (A^*)^T$
- In analogy to the property of the transpose, $\quad (AB)^\dagger = B^\dagger A^\dagger$.
- A complex matrix with $A = A^\dagger$ is *Hermitian*.
  If $A = -A^\dagger$, the matrix is anti-Hermitian.

Example: CP3 September 2010. No. 7

Define a Hermitian operator. Let A and B be two Hermitian operators. Which of the following operators are also Hermitian?

$$i(\mathbf{AB}-\mathbf{BA}), \quad (\mathbf{AB}-\mathbf{BA}), \quad \frac{1}{2}(\mathbf{AB}+\mathbf{BA})$$

If C is a non-Hermitian operator, is the product $C^\dagger C$ Hermitian? [6]

Definition of Hermitian operator: $A^\dagger = A$ where $A^\dagger = (A^T)^*$

a) Hermitian of $i(AB - BA)$
$[i(AB-BA)]^\dagger = -i(B^\dagger A^\dagger - A^\dagger B^\dagger) = -i(BA-AB) = i(AB-BA)$
   - YES Hermitian

b) $(AB-BA)^\dagger = (B^\dagger A^\dagger - A^\dagger B^\dagger) = (BA-AB) = -(AB-BA)$
   - NO not Hermitian

c) $\frac{1}{2}(AB+BA)^\dagger = \frac{1}{2}(B^\dagger A^\dagger + A^\dagger B^\dagger) = \frac{1}{2}(BA+AB)$
   - YES Hermitian

d) Hermitian of $C^\dagger C$
$(C^\dagger C)^\dagger = C^\dagger C^{\dagger\dagger} = C^\dagger C$
   - YES Hermitian

e) Inverse of a matrix

- For a matrix A, the *inverse* of the matrix A^{-1} is such that: $AA^{-1} = A^{-1}A = I$.
  - In analogy to the property of the transpose, $(AB)^{-1} = B^{-1}A^{-1}$
- A real matrix with $A^T = A^{-1}$ is *orthogonal*,
- A matrix with $A^\dagger = A^{-1}$ is *unitary*,
- A matrix with $AA^\dagger = A^\dagger A$ is *normal*, commutes with its Hermitian congugate.

e) Trace of an n $\times$ n matrix

- Defined as the sum of diagonal elements (the matrix must be square):

$$\text{Tr}\, A = A_{11} + A_{22} + \cdots + A_{nn} = \sum_{i=1}^{n} A_{ii}$$

- Can easily show
  - $\text{Tr}(A \pm B) = \text{Tr}A \pm \text{Tr}B$
  - $\text{Tr}(ABC) = \text{Tr}(CAB) = \text{Tr}(BCA)$ (cyclic permutations).

e) Rank of an m $\times$ n matrix

- The *rank* of an $m \times n$ matrix is defined as the number of *linear independent* rows or columns in the matrix (whichever is the smallest).

- An alternative definition: the rank of an $m \times n$ matrix is equal to the size of the largest *square* sub-matrix that is contained in the m rows and n columns of the matrix whose determinant is non-zero.

Matrix operations

- Matrix summation $C = A + B, \rightarrow \; C_{ij} = (A + B)_{ij} = A_{ij} + B_{ij}$
- Multiplication by a scalar $\rightarrow \; \lambda A_{ij} = (\lambda A)_{ij}$
- Matrix multiplication $C = A.B$
  $C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{in}B_{nj}$
  i.e. $C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$ for all $i = 1$ to m and all $j = 1$ to p.
  Note that $AB \neq BA$

4. Determinants and matrix inverses

Evaluating a general N $\times$ N determinant

- For an N $\times$ N matrix A, for each element A_{ij} we define a *minor* M_{ij}
- M_{ij} is the determinant of the (N-1) $\times$ (N-1) matrix obtained from A by deleting row i and column j.
- We also define *cofactor* $C_{ij} = (-1)^{(i+j)} M_{ij}$ (the "signed" minor of the same element).
- The determinant is then defined as the sum of the products of the elements of any row or column with their corresponding cofactors.

i.e. $det(A) = \sum_{j=1}^{N} A_{mj} C_{mj} = \sum_{i=1}^{N} A_{ik} C_{ik}$

for *ANY* row m or column k.

Useful properties of determinants

- If we interchange 2 *adjacent* rows or 2 *adjacent* columns of A to give B, then $det(B) = -det(A)$
- $det(A^T) = det(A)$
- $det(AB) = det(BA) = det(A) \times det(B)$
- $det(A^{-1}) = 1/det(A)$
- If B results from multiplying one row or column of A by a scalar λ then $det(B) = \lambda \times det(A)$
- For a matrix A where two or more rows (or columns) are equal or linearly dependent, then $det(A) = 0$
- If B results from adding a multiple of one row to another row, or a multiple of one column to another column, then $det(B) = det(A)$ (determinant unchanged).

Example: CP3 September 2007, No. 2

A is a non-singular 3×3 matrix and $B = 2A^{-1}$. Calculate Tr(AB) and det(A)det(B). *[4]*

- $Tr(AB) = Tr(2AA^{-1}) = 2Tr(I) = 6$
- $det(A) \times det(B) = det(A) \times det(2A^{-1})$
  $= det(2AA^{-1}) = det(2I) = 2^3 = 8$

Inverse of a matrix

For a square matrix A: $\quad AA^{-1} = A^{-1}A = I$

Prescription to find A^{-1}:

1. Start from a square matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad (2)$$

2. Form the matrix of cofactors of A:

$$C = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix} \quad (3)$$

   where cofactor $C_{ij} = [\text{minor}] \times [\text{sign}] = \mathrm{M_{ij}} \times (-1)^{(\mathrm{i+j})}$ as before.

3. Take the transpose $C \Rightarrow C^T$ (the adjugate matrix)

4. Divide by the determinant of A.

Then the elements of A^{-1} are $\quad (A^{-1})_{ik} = (C^T)_{ik}/|A| = C_{ki}/|A|$

If $|A| = 0$, the matrix the matrix has no inverse (i.e. singular).

Example: CP3 June 2010. No. 1

Determine whether the following matrices are orthogonal, unitary, hermitian, or none of these (note that some may be more than one):

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & 0 & -\sqrt{3} \\ 0 & 1 & 0 \\ \sqrt{3} & 0 & 1 \end{pmatrix}$$ [5]

- 3 definitions and 9 potential tests
- $A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, A^T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, A^\dagger = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = A \rightarrow$ Hermitian
- $A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, A^{-1} = \frac{1}{(-1)}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = A^\dagger \rightarrow$ Unitary
- $A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad A^{-1} \neq A^T \rightarrow$ NOT orthogonal
- Similarly $A = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \rightarrow$ Unitary, NOT Hermitian, NOT orthogonal

CP3 June 2010. No. 1, continued

- $\frac{1}{2}\begin{pmatrix} 1 & 0 & -\sqrt{3} \\ 0 & 1 & 0 \\ \sqrt{3} & 0 & 1 \end{pmatrix}, \quad A^T = \frac{1}{2}\begin{pmatrix} 1 & 0 & \sqrt{3} \\ 0 & 1 & 0 \\ -\sqrt{3} & 0 & 1 \end{pmatrix},$

  $A^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 0 & \sqrt{3} \\ 0 & 1 & 0 \\ -\sqrt{3} & 0 & 1 \end{pmatrix} \neq A \quad \rightarrow$ NOT Hermitian
- Determinant: $(\frac{1}{2})^3 \begin{vmatrix} 1 & 0 & -\sqrt{3} \\ 0 & 1 & 0 \\ \sqrt{3} & 0 & 1 \end{vmatrix} \quad \rightarrow \; |A| = \frac{1}{8}(1 - 0 + 3) = \frac{1}{2}$
- Matrix inverse, get matrix of co-factors:

  $C = \frac{1}{4}\begin{pmatrix} 1 & 0 & -\sqrt{3} \\ 0 & 4 & 0 \\ \sqrt{3} & 0 & 1 \end{pmatrix}, \quad C^T = \frac{1}{4}\begin{pmatrix} 1 & 0 & \sqrt{3} \\ 0 & 4 & 0 \\ -\sqrt{3} & 0 & 1 \end{pmatrix}$

  $A^{-1} = C^T/|A| = \frac{1}{2}\begin{pmatrix} 1 & 0 & \sqrt{3} \\ 0 & 4 & 0 \\ -\sqrt{3} & 0 & 1 \end{pmatrix} \quad \neq A^\dagger \quad \rightarrow$ NOT unitary
- $A^{-1} \neq A^T \quad \rightarrow$ NOT orthogonal