CP1 REVISION LECTURE 3

INTRODUCTION TO CLASSICAL MECHANICS

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OUTLINE : CP1 REVISION LECTURE 3 : INTRODUCTION TO CLASSICAL MECHANICS

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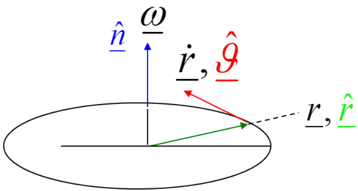
1. Angular velocity and angular acceleration

- Definition of angular velocity for rotation in a circle

$$\underline{\dot{\mathbf{r}}} = \underline{\omega} \times \underline{\mathbf{r}}$$



- Angular acceleration:

$$\underline{\alpha} = \underline{\dot{\omega}}$$

- Definition of the Moment of Inertia of the system I of particles or body rotating about a common axis of symmetry (cf. $\underline{\mathbf{p}} = m\underline{\mathbf{v}}$ for a linear system)

$$\underline{\mathbf{J}} = \mathrm{I}\underline{\omega}$$ where $$\mathrm{I} = \sum_i m_i r_i^2$$

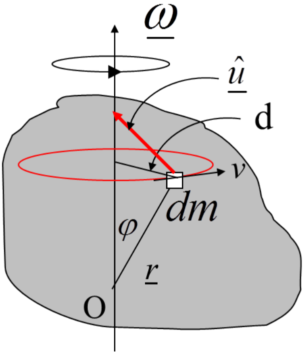
- Torque associated with the rotation

$$\underline{\tau} = \frac{d}{dt}\underline{\mathbf{J}} = \mathrm{I}\underline{\alpha}$$

2. The Moment of Inertia
Calculation of moment of inertia of rigid body

A rigid body may be considered as a collection of infinitesimal point particles whose relative distance does not change during motion.

- Mass = $\int dm$, *where* $dm = \rho dV$ and ρ is the volume density
- $\mathtt{I} = \left(\sum_i^N m_i d_i^2\right) \rightarrow \int_V d^2 \rho dV$ where d is the perpendicular distance to the axis of rotation.
- This integral gives the moment of inertia about the axis of rotation.



Rotation about a principal axis

- In general $\underline{\mathbf{J}} = \widetilde{\mathtt{I}}\,\underline{\omega}$, where $\widetilde{\mathtt{I}}$ is the *Moment of Inertia Tensor*

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \mathtt{I}_{xx} & \mathtt{I}_{xy} & \mathtt{I}_{xz} \\ \mathtt{I}_{yx} & \mathtt{I}_{yy} & \mathtt{I}_{yz} \\ \mathtt{I}_{zx} & \mathtt{I}_{zy} & \mathtt{I}_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

- Whenever possible, one aligns the axes of the coordinate system in such a way that the mass of the body evenly distributes around the axes: we choose *axes of symmetry*.

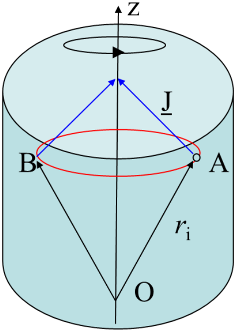
$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \mathtt{I}_x & 0 & 0 \\ 0 & \mathtt{I}_y & 0 \\ 0 & 0 & \mathtt{I}_z \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

The diagonal terms are called the *principal axes* of the moment of inertia.

- Whenever we rotate about an axis of symmetry, for every point A there is a point B which cancels it, and

$$\underline{\mathbf{J}} \rightarrow J_z\,\hat{\underline{\mathbf{z}}} = \mathtt{I}_z\,\omega_z\,\hat{\underline{\mathbf{z}}}$$

and where $\underline{\mathbf{J}}$ is parallel to $\underline{\omega}$ along the $\underline{z}$ axis



Moment of inertia & energy of rotation

Particles rotating in circular motion about a common axis of rotation with angular velocity $\underline{\omega}$ (where $\underline{\mathbf{v}}_i = \underline{\omega} \times \underline{\mathbf{r}}_i$).

- Kinetic energy of mass m_i :

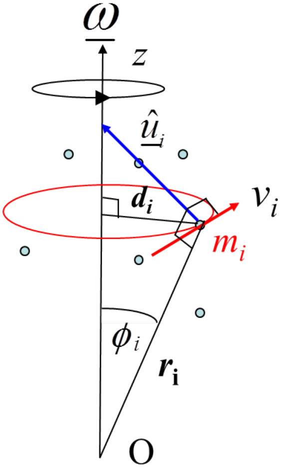
  $T_i = \frac{1}{2} m_i v_i^2$
- Total KE = $\frac{1}{2} \sum_i \left(m_i v_i^2 \right)$
- $\underline{\mathbf{v}}_i = \underline{\omega} \times \underline{\mathbf{r}}_i$

  $v_i = \omega \, r_i \, \sin \phi_i$

  $\sin \phi_i = \frac{d_i}{r_i}$
- $T_{rot} = \frac{1}{2} \left(\sum_i^N m_i \, d_i^2 \right) \omega^2$

  $T_{rot} = \frac{1}{2} \mathrm{I} \, \omega^2$

  where I is calculated about the axis of rotation



2.1 Example : MoI of a thin rectangular plate

About the x axis

- $\mathtt{I}_x = \int y^2 \, dm$

$dm = \rho \, dx \, dy \; ; \; \rho = \frac{M}{ab}$

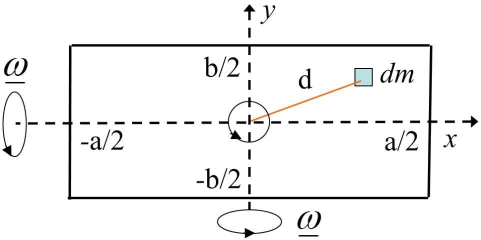
- $\mathtt{I}_x = \int y^2 \rho \, dx \, dy$

$$= \left[\rho \frac{y^3}{3}\right]_{-\frac{b}{2}}^{+\frac{b}{2}} [x]_{-\frac{a}{2}}^{+\frac{a}{2}}$$

$$= \rho \, a \left[\frac{b^3}{24} + \frac{b^3}{24}\right]$$

$$= \rho a \, b \left[\frac{b^2}{12}\right]$$

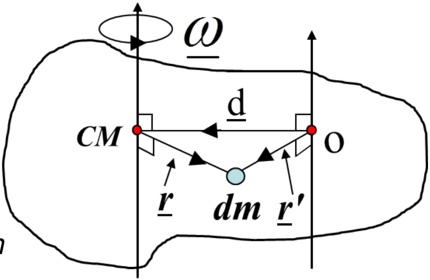
- Hence $\mathtt{I}_x = \frac{M\,b^2}{12}$



2.2 Parallel axis theorem

$\mathtt{I}_{CM}$ is the moment of inertia of body mass M about an axis passing through its centre of mass. $\mathtt{I}$ is the moment of inertia about a parallel axis a distance d from the first.

- $\mathtt{I}_{CM} = \int r^2 dm$
- $\underline{\mathbf{r}}' = \underline{\mathbf{d}} + \underline{\mathbf{r}}$
- $r'^2 = d^2 + 2\,\underline{\mathbf{d}} \cdot \underline{\mathbf{r}} + r^2$
- About the parallel axis :
  $\mathtt{I} = \int r'^2 dm$



$$= \int d^2 dm + 2\,\underline{\mathbf{d}} \cdot \underbrace{\int \underline{\mathbf{r}}\, dm}_{=0} + \int r^2 dm$$

(definition of CM)

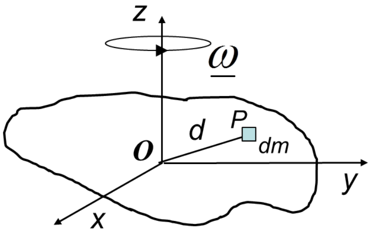
- Hence $\mathtt{I} = \mathtt{I}_{CM} + Md^2$

2.3 Perpendicular axis theorem

Consider a rigid object that *lies entirely within a plane*. The perpendicular axis theorem links $\mathtt{I}_z$ (MoI about an axis perpendicular to the plane) with $\mathtt{I}_x$, $\mathtt{I}_y$ (MoI about two perpendicular axes lying within the plane).

- Consider perpendicular axes x, y, z (which meet at origin O) ; the body lies in the xy plane
- $\mathtt{I}_z = \int d^2\, dm$
  $= \int \left(x^2 + y^2\right)\, dm$
  $= \int x^2\, dm + \int y^2\, dm$
  $\rightarrow$ $\mathtt{I}_z = \mathtt{I}_x + \mathtt{I}_y$

  This is the *perpendicular axis theorem*.



Example : compound pendulum

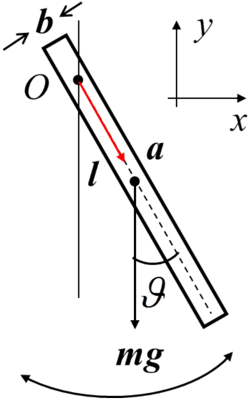
Rectangular rod length a width b mass m swinging about axis O, distance ℓ from the CM, in plane of paper

- $\mathtt{I}_x = \frac{mb^2}{12}$, $\mathtt{I}_y = m\,\frac{ma^2}{12}$
- Perpendicular axis theorem :

  $\mathtt{I}_z \equiv \mathtt{I}_{CM} = m\,(\frac{a^2+b^2}{12})$
- Parallel axis theorem :

  $\mathtt{I} = \mathtt{I}_{CM} + m\ell^2$
- Torque about O $= \underline{\mathbf{r}} \times \underline{\mathbf{F}}$:

  $\underline{\tau} = -m\,g\,\ell\sin\theta\,\hat{\underline{\mathbf{k}}}$
- $\underline{\mathbf{J}} = \mathtt{I}\,\underline{\omega} = \mathtt{I}\,\dot{\theta}\,\hat{\underline{\mathbf{k}}}$
- Differentiate wrt t : $\underline{\tau} = \mathtt{I}\,\ddot{\theta}\,\hat{\underline{\mathbf{k}}}$
- Equate $\mathtt{I}\,\ddot{\theta} = -m\,g\,\ell\sin\theta$

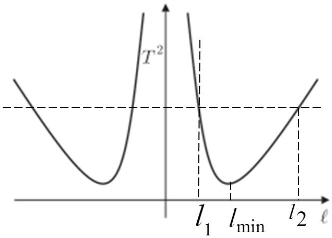


Compound pendulum continued

- $\mathtt{I}\,\ddot{\theta} = -m\,g\,\ell\sin\theta$

  where $\mathtt{I} = m\,(\frac{a^2+b^2}{12}) + m\,\ell^2$
- Small angle approximation : $\ddot{\theta} + \frac{m\,g\,\ell}{\mathtt{I}}\theta = 0$
- SHM with period $T = 2\pi\sqrt{\frac{\mathtt{I}}{mg\ell}}$

  $\rightarrow \quad T = 2\pi\sqrt{\frac{a^2+b^2+12\ell^2}{12g\ell}}$



Example : solid ball rolling down slope

[Energy of ball] = [Rotational KE in CM] + [KE of CM] + [PE]

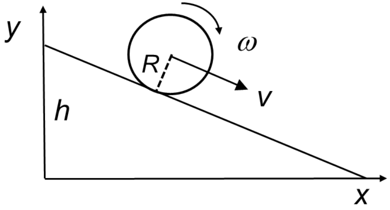
$$E = \frac{1}{2}\mathrm{I}\omega^2 + \frac{1}{2}Mv^2 + Mgy$$

- Ball falls a distance h from rest $\rightarrow$ at $y = 0$:

$$M\,g\,h = \frac{1}{2}\mathrm{I}\omega^2 + \frac{1}{2}Mv^2$$
$$= \frac{1}{2}\mathrm{I}\left(\frac{v}{R}\right)^2 + \frac{1}{2}Mv^2$$

- Solid sphere: $\mathrm{I} = \frac{2}{5}M\,R^2$
- $Mgh = \frac{1}{2}Mv^2\left(\frac{2}{5} + 1\right)$

$$\rightarrow \quad v = \sqrt{\frac{10}{7}gh}$$

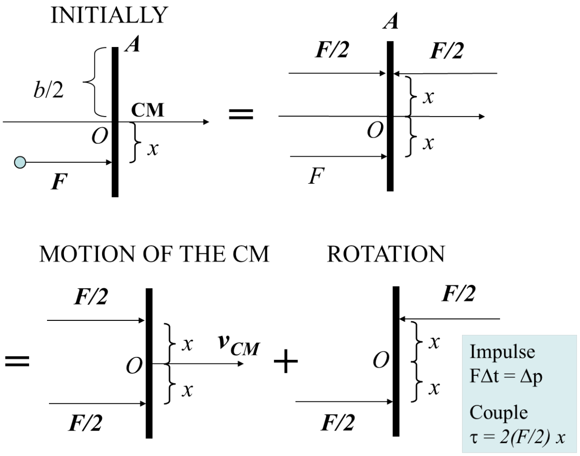


Compare with a solid cylinder $\mathrm{I} = \frac{1}{2}M\,R^2 \rightarrow v = \sqrt{\frac{4}{3}gh}$

The ball gets to the bottom faster !

Example : A rod receives an impulse

A rectangular rod receives an impulse from a force distance x from its centre of mass. Describe the subsequent motion.



A rod receives an impulse, continued

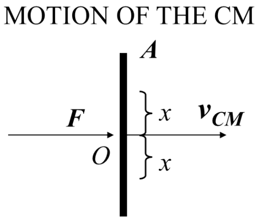
- Impulse ($\Delta p = F\Delta t$) at point x from its centre of mass
- Force applied to the CM : $F = ma$
- Moment of inertia wrt CM : $\mathtt{I}_{CM} = \frac{1}{12}Mb^2$
- Torque (couple) about O $= \mathtt{I}_{CM}\,\ddot{\theta}$
- Hence $\mathtt{I}_{CM}\,\ddot{\theta} = \frac{Mb^2}{12}\ddot{\theta} = x \times \frac{F}{2} \times 2$

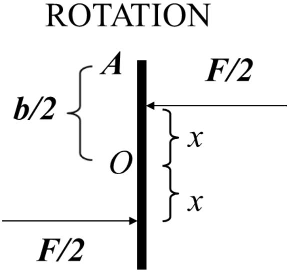
  Rotational motion $\rightarrow \ddot{\theta} = \frac{12Fx}{Mb^2}$
- Acceleration at A due to rotation

$$a_{rot} = -\frac{b}{2}\ddot{\theta}$$

- Acceleration at A due to translation

$$a_{trans} = \frac{F}{m}$$





3 . Lagrangian Mechanics

The Lagrangian : $L = T - U$

- In 1D : Kinetic energy $T = \frac{1}{2}m\dot{x}^2$ No dependence on x

  Potential energy $U = U(x)$ No dependence on $\dot{x}$
- The Lagrangian in 1D : $L = \frac{1}{2}m\dot{x}^2 - U(x)$
- $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$ and $\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x}$ gives force F
- Differential wrt t : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$
- Hence we get the Euler - Lagrange equation for x :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

- Now generalize : the Lagrangian becomes a function of $2n$ variables (n is the dimension of the configuration space). Variables are the positions and velocities

  $L(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}$$

Definitions

- Generalised coordinates : A set of parameters $q_k(t)$ that specifies the system configuration. q_k may be a geometrical parameter, x, y, z, a set of angles $\cdots$ etc
- Degrees of Freedom : The number of independent coordinates that is sufficient to describe the configuration of the system uniquely.
- Constraints : These are imposed when its components are not permitted to move freely in 3-D.
- Conjugate (generalized) momentum : $p_k = \frac{\partial L}{\partial \dot{q}_k}$
  Following on : E-L equation then reads $\dot{p}_k = \frac{\partial L}{\partial q_k}$
- Cyclic (or ignorable) coordinate q_k : If the Lagrangian L does not explicitly depend on q_k $-$ then in this case
  $\frac{\partial L}{\partial q_k} = 0$ and $p_k = \frac{\partial L}{\partial \dot{q}_k}$ = constant

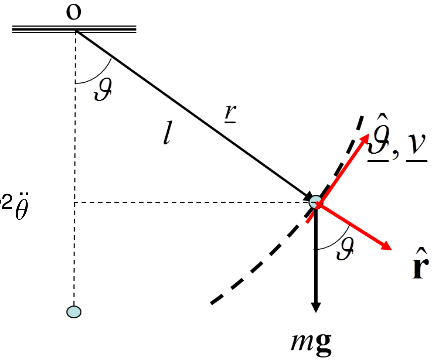
The momentum conjugate to a cyclic coordinate is a constant of motion

Example : simple pendulum

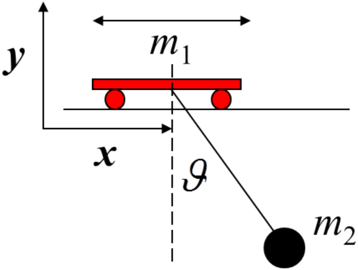
Evaluate simple pendulum using Euler-Lagrange equation

- Single variable $q_k \to \theta$
- $v = \ell\,\dot{\theta}$
- $T = \frac{1}{2} m\ell^2\dot{\theta}^2$
- $U = -mg\ell\cos\theta$
- $L = \frac{1}{2} m\ell^2\dot{\theta}^2 + mg\ell\cos\theta$
- $\frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta} \;\to\; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\ell^2\ddot{\theta}$
- $\frac{\partial L}{\partial \theta} = -mg\ell\sin\theta$
- E-L $\to m\ell^2\ddot{\theta} + mg\ell\sin\theta = 0$

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$



Pendulum on a trolley



- Pendulum's pivot can now move freely in x direction
- Pivot coordinates : $(x, 0)$
- Pendulum coordinates : $(x + \ell \sin\theta, -\ell\cos\theta)$

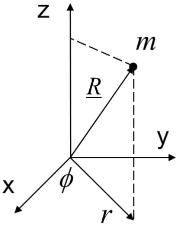
$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\frac{d}{dt}(x + \ell\sin\theta)\right)^2 + \\ + \frac{1}{2}m_2\left(\frac{d}{dt}(-\ell\cos\theta)\right)^2$$

$$U = -m_2 g\ell\cos\theta$$

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2\ell^2\dot{\theta}^2 + \\ + m_2\ell\dot{x}\cos\theta\dot{\theta} + m_2 g\ell\cos\theta$$

- $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{\partial L}{\partial\theta} \rightarrow \ddot{x}\cos\theta + \ddot{\theta}\ell + g\sin\theta = 0$
- $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right) = \frac{\partial L}{\partial x} \rightarrow \ddot{x}(m_1 + m_2) - m_2\ell\dot{\theta}^2\sin\theta + \ddot{\theta}m_2\ell\cos\theta = 0$
- Small angle approx and solve $\rightarrow$ $\ddot{\theta} + \frac{(m_1+m_2)}{m_1}\frac{g}{\ell}\theta = 0$

3.1 The Lagrangian in various coordinate systems



Cylindrical coords

- Cartesian coordinates

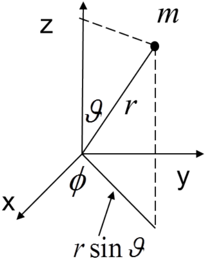
  $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$

- Cylindrical coordinates

  $L = \frac{1}{2}m(\dot{r}^2 + r^2\,\dot{\phi}^2 + \dot{z}^2) - U(r, \phi, z)$

- Spherical coordinates

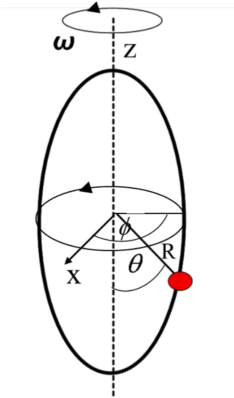
$L = \frac{1}{2}m(\dot{r}^2 + r^2\,\dot{\theta}^2 + (r\sin\theta)^2\,\dot{\phi}^2) - U(r, \theta, \phi)$



Spherical coords

3.2 Example : bead on rotating hoop

A vertical circular hoop of radius R rotates about a vertical axis at a constant angular velocity ω. A bead of mass m can slide on the hoop without friction. Describe the motion of the bead.



- Use spherical coordinates :

  $$T = \frac{1}{2}m\left(\dot{R}^2 + R^2\dot{\theta}^2 + (R\sin\theta)^2\dot{\phi}^2\right)$$

- But $\dot{R} = 0$, $\dot{\phi} = \omega =$ constant

  $$T = \frac{1}{2}m(R^2\,\dot{\theta}^2 + (R\sin\theta)^2\,\omega^2)$$

- $U = -mgR\cos\theta$
- $L = T - U$

  $$L = \frac{1}{2}m(R^2\,\dot{\theta}^2 + (R\sin\theta)^2\,\omega^2) + mgR\cos\theta$$

  One single generalized coordinate : θ

Bead on rotating hoop, continued

$$L = \frac{1}{2} m(R^2\,\dot{\theta}^2 + R^2 \sin^2\theta\,\omega^2) + mgR\cos\theta$$

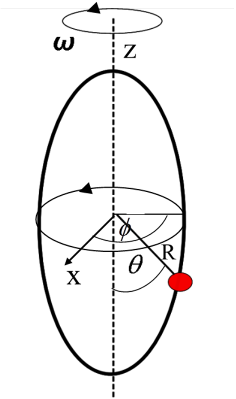
- E-L equation: $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{\partial L}{\partial\theta}$
- $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{d}{dt}\left(m\,R^2\dot{\theta}\right) = m\,R^2\ddot{\theta}$

  $\frac{\partial L}{\partial\theta} = m\,R^2 sin\,\theta\cos\theta\,\omega^2 - mgR\sin\theta$

  $\rightarrow\ \ddot{\theta} = sin\,\theta\cos\theta\,\omega^2 - \frac{g}{R}\sin\theta$

  $\rightarrow\ \ddot{\theta} + \left(\omega_0^2 - \omega^2\cos\theta\right)\sin\theta = 0$

  where $\omega_0^2 = \frac{g}{R}$



- If $\omega = 0$, $\ddot{\theta} + \omega_0^2\sin\theta = 0\ \rightarrow$ SHM, back to pendulum formula

4. The Hamiltonian

- Conjugate momentum : $p_k = \frac{\partial L}{\partial \dot{q}_k}$, from E-L $\dot{p}_k = \frac{\partial L}{\partial q_k}$
- Define Hamiltonian $H = \sum_k p_k \dot{q}_k - L$
- Can show (see HT lectures) $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$
- If L does not depend *explicitly* on time, H is a constant of motion
- Take kinetic energy $T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$
- $L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) - U(x, y, z)$
- $H = \sum_k p_k \dot{q}_k - L = m \left(\dot{x}.\dot{x} + \dot{y}.\dot{y} + \dot{z}.\dot{z} \right) - (T - U)$

  $= 2T - (T - U) = T + U = E \quad \rightarrow$ total energy
- If L does not depend *explicitly* on time $\frac{dH}{dt} = 0$

  $\rightarrow$ energy is a constant of the motion

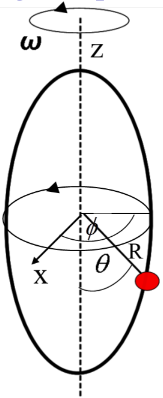
4.1 Example: re-visit bead on rotating hoop

First take the case of a free (undriven) system

- $L = \frac{1}{2} m (R^2 \dot{\theta}^2 + R^2 \sin^2\theta\, \dot{\phi}^2) + mgR\cos\theta$
- $H = \sum_k p_k \dot{q}_k - L \;\; ; \;\; p_k = \frac{\partial L}{\partial \dot{q}_k}$
- $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m R^2 \dot{\theta} \;\; ; \; p_\phi = mR^2 \sin^2\theta\, \dot{\phi}$
- $H = m R^2 \dot{\theta}^2 + mR^2 \sin^2\theta\, \dot{\phi}^2 - L$

$$= \frac{1}{2} m \left(R^2 \dot{\theta}^2 + R^2 \sin^2\theta\, \dot{\phi}^2 \right) - mgR\cos\theta$$

$$\rightarrow \; H = T + U = E$$



L does not depend explicitly on t,
H, E conserved $\rightarrow$ Hamiltonian gives the total energy

- Note: $\frac{\partial L}{\partial \dot{\phi}} = m(R\sin\theta)^2 \dot{\theta}$: which is angular momentum about O. ϕ is CYCLIC $\rightarrow \frac{\partial L}{\partial \phi} = 0 \rightarrow$ A.M conserved.

Example continued

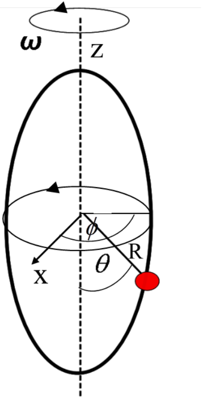
The system is now DRIVEN - hoop rotating at constant angular speed ω

- $L = \frac{1}{2} m \left(R^2 \dot{\theta}^2 + R^2 \omega^2 \sin^2 \theta \right) + mgR \cos \theta$
- $H = \sum_k p_k \dot{q}_k - L \; ; \; p_k = \frac{\partial L}{\partial \dot{q}_k}$
- $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m R^2 \dot{\theta}$; a single coordinate θ
- $H = m R^2 \dot{\theta}^2 - L \quad (\text{Note } p_\phi = \frac{\partial L}{\partial \dot{\phi}} = 0)$

  $= \frac{1}{2} m \left(R^2 \dot{\theta}^2 - R^2 \omega^2 \sin^2 \theta \right) - mgR \cos \theta$
- $E = \frac{1}{2} m \left(R^2 \dot{\theta}^2 + R^2 \omega^2 \sin^2 \theta \right) - mgR \cos \theta$

  Hence $E = H + m(R^2 \omega^2 \sin^2 \theta)$

  $\rightarrow \; E = T + U \neq H$



- $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$
- H is a constant of the motion, E is not const.

In this case the hoop has been forced to rotate at an angular velocity ω. External energy is being supplied to the system.