CP1 REVISION LECTURE 2

INTRODUCTION TO CLASSICAL MECHANICS

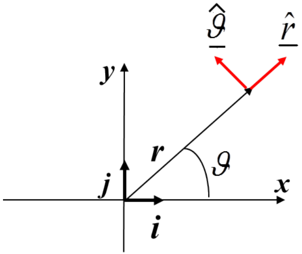
Prof. N. Harnew
University of Oxford

TT 2017

OUTLINE : CP1 REVISION LECTURE 2 : INTRODUCTION TO CLASSICAL MECHANICS

1. Angular variables

- $\underline{\hat{\mathbf{r}}} = (\underline{\mathbf{i}}\cos\theta + \underline{\mathbf{j}}\sin\theta)$ is a unit vector in the direction of $\underline{\mathbf{r}}$
- $\underline{\hat{\theta}} = (-\underline{\mathbf{i}}\sin\theta + \underline{\mathbf{j}}\cos\theta)$ is a unit vector perpendicular to $\underline{\mathbf{r}}$



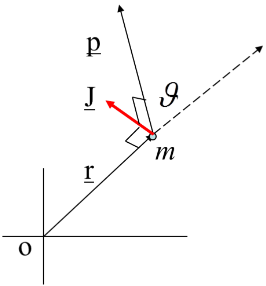
- $\underline{\mathbf{r}} = r(\underline{\mathbf{i}}\cos\theta + \underline{\mathbf{j}}\sin\theta)$
- Differentiating wrt time :

$$\underline{\mathbf{v}} = \underline{\dot{\mathbf{r}}} = \dot{r}\,\underline{\hat{\mathbf{r}}} + r\,\dot{\theta}\,\underline{\hat{\theta}}$$

- $\underline{\mathbf{a}} = \underline{\dot{\mathbf{v}}} = \underline{\ddot{\mathbf{r}}}$
- Differentiating wrt time again :

$$\underline{\mathbf{a}} = \underline{\ddot{\mathbf{r}}} = (\ddot{r} - r\,\dot{\theta}^2)\,\underline{\hat{\mathbf{r}}} + (2\dot{r}\,\dot{\theta} + r\,\ddot{\theta})\,\underline{\hat{\theta}}$$

1.1 Angular momentum and torque



- The definition of *angular momentum* for a single particle wrt origin O:

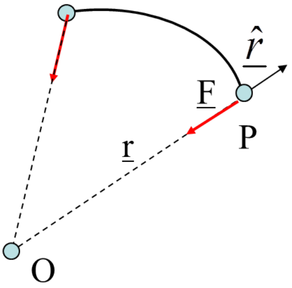
  $$\underline{\mathbf{J}} = \underline{\mathbf{r}} \times \underline{\mathbf{p}}$$

- Differentiate: $\frac{d\underline{\mathbf{J}}}{dt} = \underline{\mathbf{r}} \times \frac{d\underline{\mathbf{p}}}{dt} + \frac{d\underline{\mathbf{r}}}{dt} \times \underline{\mathbf{p}}$
- $\frac{d\underline{\mathbf{J}}}{dt} = \underline{\mathbf{r}} \times \underline{\mathbf{F}} + \underline{\mathbf{v}} \times \underline{\mathbf{p}}$ $\leftarrow$ this term$= m\underline{\mathbf{v}} \times \underline{\mathbf{v}} = 0$
- Define *torque* $\underline{\tau} = \underline{\mathbf{r}} \times \underline{\mathbf{F}} = \frac{d\underline{\mathbf{J}}}{dt}$

  (cf. Linear motion $\underline{\mathbf{F}} = \frac{d\underline{\mathbf{p}}}{dt}$)

2. Central forces

- Central force: $\underline{\mathbf{F}}$ acts towards origin (line joining O and P) always.



- $\underline{\mathbf{F}} = f(r)\,\underline{\hat{\mathbf{r}}}$ only

- Examples:

  Gravitational force $\underline{\mathbf{F}} = -\frac{GmM}{r^2}\underline{\hat{\mathbf{r}}}$

  Electrostatic force $\underline{\mathbf{F}} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\underline{\hat{\mathbf{r}}}$

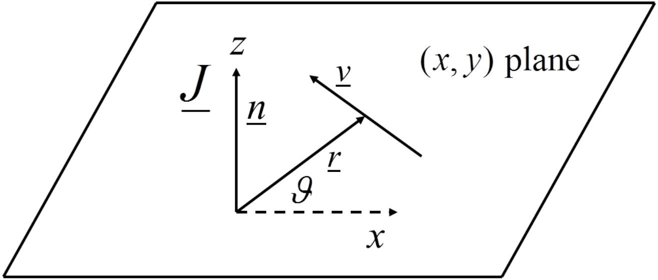
- Torque about origin : $\underline{\tau} = \underline{\mathbf{r}} \times \underline{\mathbf{F}}$
- For a central force, $\underline{\tau} = \frac{d\mathbf{J}}{dt} = \underline{\mathbf{r}} \times \underline{\mathbf{F}} = 0$

  Hence angular momentum is a constant of the motion

- $\underline{\mathbf{J}} = (mr^2\,\dot{\theta})\,\underline{\hat{\mathbf{n}}}$ = constant

Motion under a central force

- $\underline{\mathbf{J}} = m\,\underline{\mathbf{r}} \times \underline{\mathbf{v}}$
- Angular momentum is always perpendicular to $\underline{\mathbf{r}}$ and $\underline{\mathbf{v}}$
- $\underline{\mathbf{J}}$ is a constant vector ; $\underline{\mathbf{J}} \cdot \underline{\mathbf{r}} = 0$; $\underline{\mathbf{J}} \cdot \underline{\mathbf{v}} = 0$

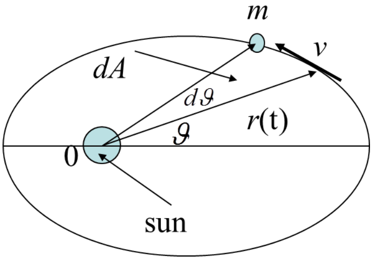


Motion under a central force lies in a plane

Sweeping out equal area in equal time

- Central force example : planetary motion : $F_r = \frac{GMm}{r^2}$
- Angular momentum is conserved

  $\rightarrow \ |\underline{\mathbf{J}}| = mr^2\,\dot{\theta} = \text{constant}$



- $dA \approx \frac{1}{2}\, r^2\, d\theta$
- $\frac{dA}{dt} = \frac{1}{2}\, r^2\, \dot{\theta}$

  $\frac{dA}{dt} = \frac{J}{2m} = \text{constant}$ (Kepler 2^{nd} Law)

Orbit sweeps out equal area in equal time

2.2 Central force : the total energy

- Total energy = kinetic + potential :
  $E = T + U(r) = \frac{1}{2}mv^2 + U(r)$ = constant
- $\underline{\mathbf{v}} = \dot{r}\underline{\hat{\mathbf{r}}} + r\dot{\theta}\underline{\hat{\theta}}$
  $\rightarrow \; |\underline{\mathbf{v}}|^2 = v_r^2 + v_\theta^2 = \dot{r}^2 + r^2\dot{\theta}^2$ (since $\underline{\hat{\mathbf{r}}} \cdot \underline{\hat{\theta}} = 0$)
- $E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + U(r)$
- No external torque: angular momentum is conserved
  $\rightarrow \; |\underline{\mathbf{J}}| = mr^2\,\dot{\theta} =$ constant

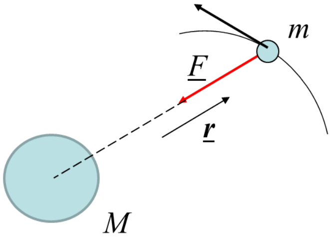
$$E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + U(r)$$

The potential term (inverse square interaction)

- $\underline{\mathbf{F}} = -\frac{A}{r^2}\,\underline{\hat{\mathbf{r}}} \;\rightarrow\; f(r) = -\frac{A}{r^2}$
- $U(r) = -\int_{r_{ref}}^{r} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}}$

  Usual to define $U(r) = 0$ at $r_{ref} = \infty$

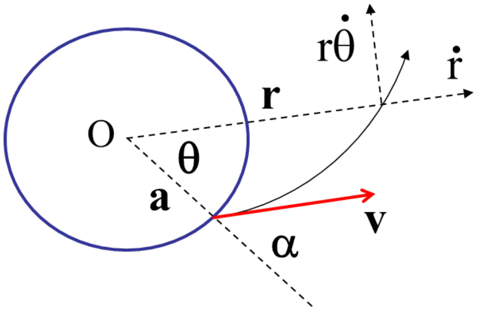
  $\rightarrow\; U(r) = -\frac{A}{r}$



Newton law of gravitation : $\underline{\mathbf{F}} = -\frac{GMm}{r^2}\,\underline{\hat{\mathbf{r}}} \;\rightarrow\; U(r) = -\frac{GMm}{r}$

Example

A projectile is fired from the earth's surface with speed *v* at an angle α to the radius vector at the point of launch. Calculate the projectile's subsequent maximum distance from the earth's surface. Assume that the earth is stationary and its radius is *a*.



Example : solution

- $U(r) = -\frac{GMm}{r}$
- $|\underline{\mathbf{J}}| = m|\underline{\mathbf{r}} \times \underline{\mathbf{v}}| = mav\sin\alpha$
- Energy equation : $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + U(r)$

  $\rightarrow\ E = \frac{1}{2}m\dot{r}^2 + \frac{ma^2v^2\sin^2\alpha}{2r^2} - \frac{GMm}{r}$
- At $r = a$: $E = \frac{1}{2}mv^2 - \frac{GMm}{a}$. At maximum height : $\dot{r} = 0$

  $\rightarrow\ \frac{1}{2}mv^2 - \frac{GMm}{a} = \frac{ma^2v^2\sin^2\alpha}{2r_{max}^2} - \frac{GMm}{r_{max}} \qquad (1)$

  $\rightarrow\ \left(v^2 - \frac{2GM}{a}\right) r_{max}^2 + 2GM\, r_{max} - a^2v^2\sin^2\alpha = 0$
- Solve and take the positive root
- Note from Equ.(1) : When $\dot{r} \rightarrow 0$ as $r_{max} \rightarrow \infty$, the rocket *just* escapes the earth's gravitational field

  i.e. $\frac{1}{2}mv^2 - \frac{GMm}{a} \rightarrow 0$, $v_{esc} = \sqrt{\frac{2GM}{a}}$ (independent of α)

3. Effective potential

- Energy equation : $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + U(r)$
- Define *effective potential* : $U_{eff}(r) = \frac{J^2}{2mr^2} + U(r)$

  $\rightarrow$ then $E = \frac{1}{2}m\dot{r}^2 + U_{eff}(r)$
- Note this has the same form as a 1-D energy expression :
  $\rightarrow$ $E = \frac{1}{2}m\dot{x}^2 + U(x)$ and can treat like a 1D problem
- Allows to predict important features of motion without solving the radial equation

  $\rightarrow$ $\frac{1}{2}m\dot{r}^2 = E - U_{eff}(r)$ $\leftarrow$ LHS is always positive

  $\rightarrow$ $U_{eff}(r) < E$

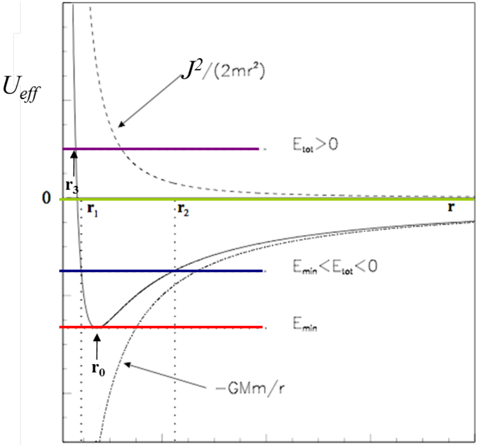
  The only locations where the particle is allowed to go are those with $U_{eff}(r) < E$

$U_{eff}(r)$ *for inverse square law*

- $U_{eff}(r) = \frac{J^2}{2mr^2} - \frac{GmM}{r}$
- $U_{eff}(r) < E_{tot}$ for all r

Three cases :

- $E_{tot} < 0$: Bound (closed) orbit with $r_1 < r < r_2$
- E_{tot} has minimum energy at $r = r_0$: $\frac{dU_{eff}}{dr} = 0$, circular motion with $\dot{r} = 0$
- $E_{tot} > 0$: Unbound (open) orbit with $r > r_3$



4. Orbits

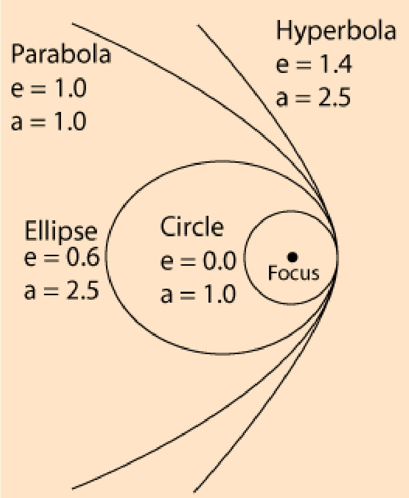
$$r(\theta) = \frac{r_0}{1+e\cos\theta}$$: the orbit equation

(Note that the derivation of this is off syllabus.)

Total energy in ellipse parameters $E = \frac{\alpha}{2r_0}\left(e^2 - 1\right)$

Also $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{\alpha}{r}$

- $e = 0, r = r_0, E = -\frac{\alpha}{2r_0}$
  $\rightarrow$ motion in a circle
- If $0 < e < 1$, $E < 0$
  $\rightarrow$ motion is an ellipse
- If $e = 1$, $E = 0$
  $r(\theta) = \frac{r_0}{1+\cos\theta}$
  $\rightarrow$ motion is a parabola
- If $e > 1$, $E > 0$
  $\rightarrow$ motion is a hyperbola



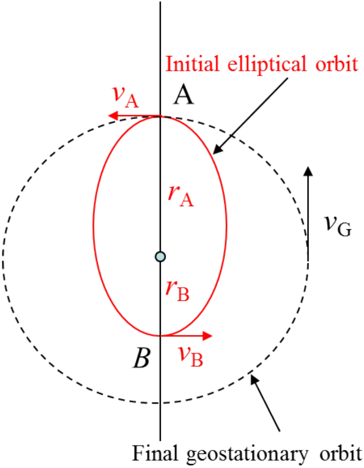
4.1 Example : putting a satellite into geostationary orbit

i) Calculate the orbital velocity in a geostationary orbit and show that its radius is approximately 40,000 km.

- Geostationary orbit : the circular orbit around the Earth above the equator which has a period of 24 hours
- Equate forces : $\frac{GMm}{r^2} = \frac{mv^2}{r} \quad \rightarrow v = \sqrt{\frac{GM}{r}}$
- Period $T = \frac{2\pi r}{v} = \frac{2\pi r^{3/2}}{(GM)^{1/2}} = 86,400$ s
- Radius $r = \left(\frac{(GM)^{1/2} T}{2\pi}\right)^{2/3} = 4.2 \times 10^7$ m
- $v_G = 3080$ ms^{-1}

ii) A satellite is to be placed in to a geostationary orbit from an elliptical orbit with perigee at a geocentric radius of $r_B =$ 8,000 km and apogee at $r_A =$42,000 km. When it is at apogee, a brief firing of its rocket motor places it into the circular orbit. Calculate the change in velocity the motor needs to provide.

Example continued



- Conservation of angular momentum, points *A* & *B*

  $v_A r_A = v_B r_B$

- Energy at *A* = energy at *B*

  $\frac{1}{2}mv_A^2 - \frac{GMm}{r_A} = \frac{1}{2}mv_B^2 - \frac{GMm}{r_B}$

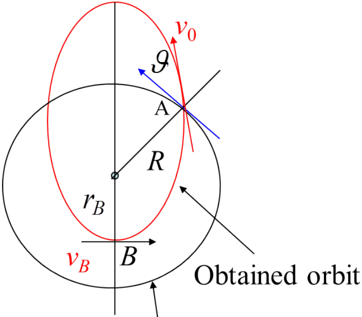
- Solve for v_A

  $\rightarrow v_A^2 = \frac{2GM(1/r_B - 1/r_A)}{(r_A^2/r_B^2 - 1)}$

- To place in geostationary orbit, need to boost rockets by $\Delta v = v_G - v_A$
- Putting in numbers $\rightarrow \Delta v = 1280 \text{ ms}^{-1}$

4.2 Example : change in orbit angular momentum

- A spacecraft is in circular motion about a planet
- The spacecraft is given an impulse which leaves the magnitude of velocity v_0 unchanged but the spacecraft now makes an angle θ wrt direction of motion
- Question: what is the apogee and perigee of the subsequent elliptical orbit?



- Conservation of angular momentum, points A & B

$$J = mv_0 R\sin(\tfrac{\pi}{2} - \theta) = mv_B r_B$$

- Energy at A = energy at B

$$\frac{1}{2}mv_0^2 - \frac{\alpha}{R} = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr_B^2} - \frac{\alpha}{r_B}$$

where $\alpha = GMm$

Example continued

- At point B, $\dot{r} = 0$, energy conservation becomes

$$\frac{1}{2}mv_0^2 - \frac{\alpha}{R} = \frac{m^2 v_0^2 R^2 \cos^2\theta}{2mr_B^2} - \frac{\alpha}{r_B}$$

- Equate forces for circular motion to get v_0 :

$$\frac{mv_0^2}{R} = \frac{\alpha}{R^2} \quad \rightarrow v_0^2 = \frac{\alpha}{mR}$$

- Sub for v_0^2 : energy conservation becomes

$$\frac{\alpha}{2R} - \frac{\alpha}{R} = \frac{\alpha R \cos^2\theta}{2r_B^2} - \frac{\alpha}{r_B}$$

- Solve for r_B $\rightarrow$ $r_B = R \pm \sqrt{R^2 - R^2\cos\theta}$

$$r_B = R(1 \pm \sin\theta)$$

(+ve solution is apogee, −ve is perigee)

4.3 Impulse leaving angular momentum unchanged

- Example: A satellite in circular orbit has been given an impulse leaving J unchanged. The kinetic energy is changed by $T = \beta T_0$. Describe the subsequent motion.
- If J is not changed, impulse must be perpendicular to the direction of motion as shown with angular part of the velocity unchanged.

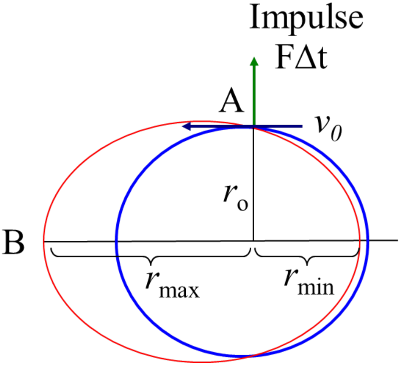
- $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{\alpha}{r}$
- Circular orbit:

  $\rightarrow \dot{r} = 0\ ,\ J = mr_0 v_0 \quad (1)$

  $\rightarrow E_{initial} = \frac{1}{2}mv_0^2 - \frac{\alpha}{r_0}$
- Equate forces :

  $\rightarrow \frac{mv_0^2}{r_0} = \frac{\alpha}{r_0^2}$

  $\rightarrow v_0^2 = \frac{\alpha}{mr_0} \qquad (2)$



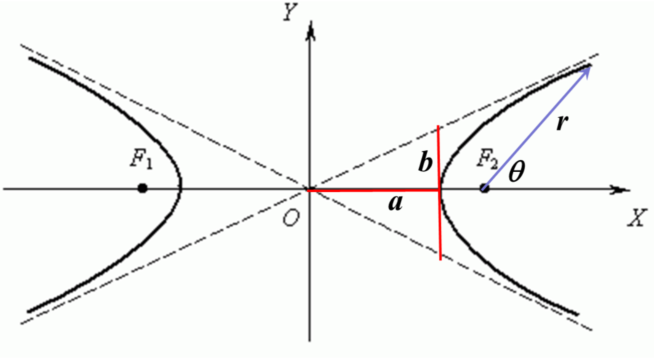
Example continued

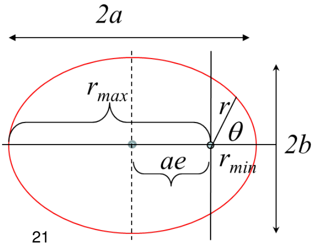
- New orbit (elliptical): $E_{new} = \frac{1}{2}\beta m v_0^2 - \frac{\alpha}{r_0}$
- Equate energies: subsequent motion described by:

$$\frac{1}{2}\beta m v_0^2 - \frac{\alpha}{r_0} = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{\alpha}{r} \qquad (3)$$

- Now solve for r_{min} , r_{max} . Set $\dot{r} = 0$ for apogee and perigee.
- From (1), (2), (3)
  $\rightarrow \quad (\beta - 2)\, r^2 + 2r_0\, r - \; r_0^2 = 0$
- $r_{min,max} = \frac{-\,r_0 \pm \sqrt{r_0^2 + (\beta - 2)\, r_0^2}}{(\beta - 2)}$

4.4 The hyperbolic orbit

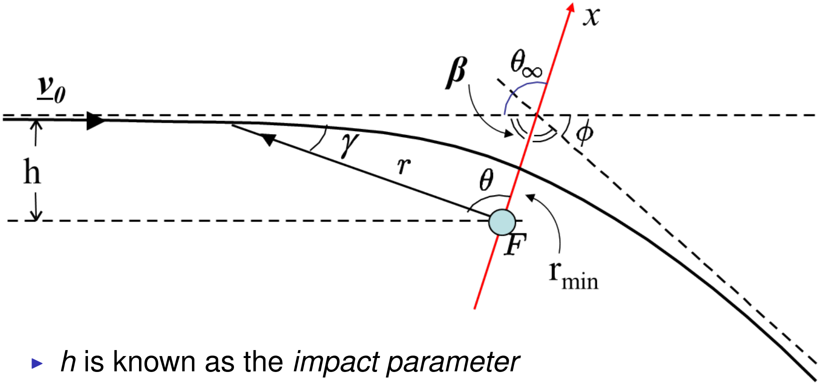




- Orbit equation : $r(\theta) = \frac{r_0}{1+e\cos\theta}$
- Ellipse : $e < 1$ $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$
- Hyperbola :
  $e > 1$ $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$

Hyperbolic orbit : the distance of closest approach

Example : a comet deviated by the gravitational attraction of a planet.
Velocity $\underline{\mathbf{v}} = \underline{\mathbf{v}}_0$ when $\underline{\mathbf{r}} \to \infty$.



- *h* is known as the *impact parameter*
- Angular momentum $\underline{\mathbf{J}} = m\,\underline{\mathbf{r}} \times \underline{\mathbf{v}}$

  $\to |\underline{\mathbf{J}}| = mvr\sin\gamma = mv_0 h$ (seen as $r \to \infty$)

  $\to$ Total energy $E = \frac{1}{2} m v_0^2$ (also seen as $r \to \infty$)

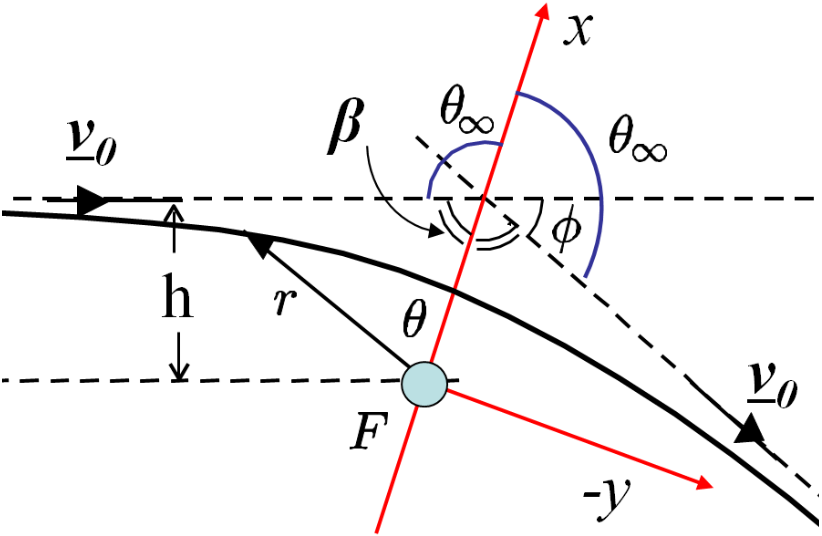
Distance of closest approach continued

- $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{\alpha}{r}$, where $\alpha = GmM$
- At distance of closest approach $r = r_{min} \rightarrow \dot{r} = 0$
- $E = \frac{J^2}{2mr_{min}^2} - \frac{\alpha}{r_{min}}$

  $\rightarrow r_{min}^2 + \frac{\alpha}{E} r_{min} - \frac{J^2}{2mE} = 0$
- Solution:
- $r_{min} = -\left(\frac{\alpha}{2E}\right)\left[1 - (1 + \frac{2EJ^2}{m\alpha^2})^{\frac{1}{2}}\right] \qquad (J^2 = (mv_0 h)^2 \; ; \; E = \frac{1}{2} mv_0^2)$
- Velocity v' at distance of closest approach: line to trajectory is a right angle.

  $\rightarrow \; J = mv' r_{min} = mv_0 h \;\; \rightarrow \;\; v' = \frac{v_0 h}{r_{min}}$

The angle of deflection, ϕ



Impulse method

- Directly from the diagram : $\Delta v_x = 2v_0 \cos\theta_\infty$ (1)
- By symmetry, integrated change in $v_y = 0$: $\Delta v_y = 0$
- Change in Δp_x : $m\Delta v_x = \int_{-\infty}^{+\infty} F_x dt = \int_{-\infty}^{+\infty} F_x \underbrace{\left(\frac{mr^2\dot{\theta}}{J}\right)}_{=1} dt$

  $\rightarrow \quad m\Delta v_x = \left(\frac{m}{J}\right) \int_{-\theta_\infty}^{+\theta_\infty} F_x r^2 d\theta$
- But $\underline{\mathbf{F}} = -\left(\frac{\alpha}{r^2}\right)\underline{\hat{\mathbf{r}}} \quad \rightarrow \quad F_x = -\frac{\alpha}{r^2}\cos\theta$

  $\rightarrow \quad m\Delta v_x = -2\left(\frac{m\alpha}{J}\right) \int_0^{\theta_\infty} \cos\theta d\theta$

  $\rightarrow \quad \Delta v_x = -\left(\frac{2\alpha}{J}\right)\sin\theta_\infty$ (2)
- From (1) & (2) $\rightarrow -\left(\frac{2\alpha}{J}\right)\sin\theta_\infty = 2v_0 \cos\theta_\infty$
- $\tan\theta_\infty = -\frac{Jv_0}{\alpha} \quad \rightarrow \quad \theta_\infty + \beta = \pi \; ; \; \phi + 2\beta = \pi$
- $\theta_\infty = \frac{\phi}{2} + \frac{\pi}{2}$; $\tan\theta_\infty = \tan\left(\frac{\phi}{2} + \frac{\pi}{2}\right) = -\cot\frac{\phi}{2}$

$$\cot\frac{\phi}{2} = \frac{J v_0}{\alpha} = \frac{mhv_0^2}{\alpha} = \frac{hv_0^2}{GM}$$