CP1 REVISION LECTURE 1

INTRODUCTION TO CLASSICAL MECHANICS

Prof. N. Harnew
University of Oxford

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OUTLINE : CP1 REVISION LECTURE 1 : INTRODUCTION TO CLASSICAL MECHANICS

1. Force and work

1.1 Newton's Laws of motion

1.2 Work done and conservative forces

2. Projectile motion

2.1 Constant acceleration

2.2 Resistive force $F_R \propto v$

2.3 Resistive force $F_R \propto v^2$

3. Rocket motion

3.1 The rocket : vertical launch

4. Two-body collisions

4.1 The Centre of Mass frame

4.2 Two-body elastic collision in 1D : Lab to CM system

4.3 Solving collision problems in the CM frame

4.4 Inelastic collisions

Outline of revision lectures

Three revision lectures:

- Today:
  - Force and work
  - Projectile motion
  - Rocket motion
  - Two-body collisions
- Tomorrow:
  - Central forces
  - Effective potential
  - Circular motion and orbits
- Tuesday Week 2:
  - Rotational motion
  - Lagrangian mechanics

1. Force and work

1.1 Newton's Laws of motion

- NI: Every body continues in a state of rest or in uniform motion (constant velocity in straight line) unless acted upon by an external force.
- NII: The rate of change of momentum is equal to the applied force: $\underline{\mathbf{F}} = m\underline{\mathbf{a}}$
- NIII: Action and reaction forces are equal in magnitude and opposite in direction.

Problems of particle motion involve solving the equation of motion in 3D:

$$\underline{\mathbf{F}} = m\frac{d\underline{\mathbf{v}}}{dt}$$

1.2 Work and conservative forces

Work done from A→ B $\quad W_{ab} = \int_a^b F\,dx = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2$

For any conservative force: $\quad W_{ab} = U(a) - U(b)$

For a conservative field of force, the work done depends only on the initial and final positions of the particle independent of the path. Equivalent definitions:

- The force is derived from a (scalar) potential function: $\underline{\mathbf{F}}(\underline{\mathbf{r}}) = -\nabla U \quad \rightarrow \quad F(x) = -\frac{dU}{dx}$ etc.
- There is zero net work by the force when moving a particle around any closed path: $W = \oint_c F dx = 0$
- In equivalent vector notation $\quad \underline{\nabla} \times \underline{\mathbf{F}} = 0$

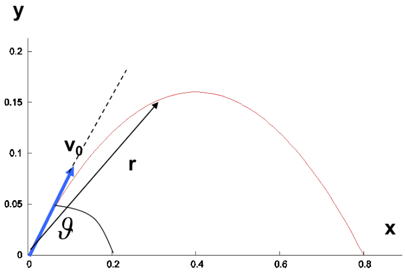
For *any* force: $\quad W_{ab} = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2$

For a conservative force: $\quad W_{ab} = U(a) - U(b)$

If these are different, energy is dissipated to the environment

2. *Projectile motion*

2.1 Constant acceleration in 2D, no resistive force



- $\underline{\mathbf{a}} = \frac{d\underline{\mathbf{v}}}{dt} = \text{constant}$
- $\int_{v_0}^{v} d\underline{\mathbf{v}} = \int_0^t \underline{\mathbf{a}}dt$

  $\rightarrow \underline{\mathbf{v}} = \underline{\mathbf{v}}_0 + \underline{\mathbf{a}}t$
- $\int_0^r d\underline{\mathbf{r}} = \int_0^t (\underline{\mathbf{v}}_0 + \underline{\mathbf{a}}t)dt$

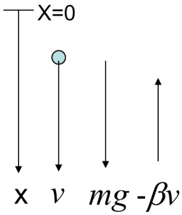
  $\rightarrow \underline{\mathbf{r}} = \underline{\mathbf{v}}_0 t + \frac{1}{2}\underline{\mathbf{a}}t^2$

Under gravity: $\underline{\mathbf{a}} = -g\underline{\hat{\mathbf{y}}} \rightarrow a_x = 0; \; a_y = -g$

- $v_x = v_0 \cos\theta$
- $v_y = v_0 \sin\theta - gt$
- $x = (v_0 \cos\theta)t$
- $y = (v_0 \sin\theta)t - \frac{1}{2}gt^2$

Trajectory: $y = (\tan\theta)x - \frac{g}{2v_0^2}(\sec^2\theta)x^2$

2.2 *Resistive force* $F_R \propto v$



- Example of body falling *vertically downwards* under gravity with air resistance $\propto$ velocity.

  $v = 0$ at $x = 0$ and $t = 0$

- Equation of motion:
  $m\frac{dv}{dt} = mg - \beta v$

- $\int_0^v \frac{dv}{g - \alpha v} = \int_0^t dt$ where $\alpha = \frac{\beta}{m}$

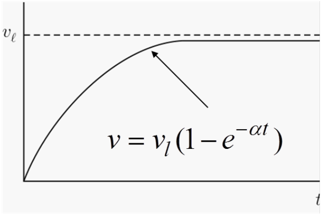
- $\left[-\frac{1}{\alpha} log_e(g - \alpha v)\right]_0^v = t$

  $\rightarrow \quad \frac{g - \alpha v}{g} = \exp\left(-\alpha t\right)$

$$v = \frac{g}{\alpha}\left(1 - \exp(-\alpha t)\right)$$

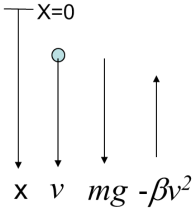
- Calculate distance travelled

  $x = \int_0^t \frac{g}{\alpha}\left(1 - \exp(-\alpha t)\right) dt$



- As $t \rightarrow \infty$, $v \rightarrow \frac{g}{\alpha}$
- Terminal velocity

2.3 Resistive force $F_R \propto v^2$



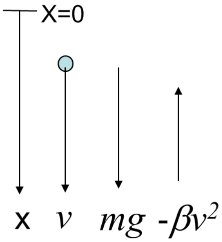
- Body falls *vertically downwards* under gravity with air resistance $\propto$ [velocity]^2, $v = 0$, $x = 0$ at $t = 0$
- Equation of motion: $m\frac{dv}{dt} = mg - \beta v^2$
- Terminal velocity when $\frac{dv}{dt} = 0$: $v_T = \sqrt{\frac{mg}{\beta}}$
  - Equation of motion becomes $\frac{dv}{dt} = g\left(1 - v^2/v_T^2\right)$
  - Integrate $\int_0^v \frac{dv}{g\left(1-v^2/v_T^2\right)} = \int_0^t dt$
  - Standard integral : $\int \frac{1}{1-z^2}\, dz = \frac{1}{2}\log_e\left(\frac{1+z}{1-z}\right)$
  - $\left[\frac{v_T}{2g} log_e\left(\frac{1+v/v_T}{1-v/v_T}\right)\right]_0^v = t \;\rightarrow\; \frac{1+v/v_T}{1-v/v_T} = \exp(t/\tau)$, where $\tau = \frac{v_T}{2g}$

    $\rightarrow \; (1 - \frac{v}{v_T}) = (1 + \frac{v}{v_T})\exp(-\frac{t}{\tau})$

Velocity as a function of time:

$$v = v_T\left[\frac{1-\exp(-t/\tau)}{1+\exp(-t/\tau)}\right]$$

Velocity as a function of distance $F_R \propto v^2$



- Equation of motion: $\frac{dv}{dt} = g\left(1 - v^2/v_T^2\right)$

- Write $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$

  - $\int_0^v \frac{v\,dv}{g\left(1-v^2/v_T^2\right)} = \int_0^x dx$

  - $\left[-\frac{v_T^2}{2g} log_e\left(1 - v^2/v_T^2\right)\right]_0^v = x$

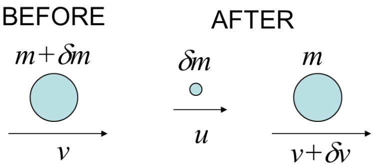
$\rightarrow \quad \left(1 - v^2/v_T^2\right) = \exp\left(-x/x_T\right) \quad , \quad$ where $x_T = \frac{v_T^2}{2g}$

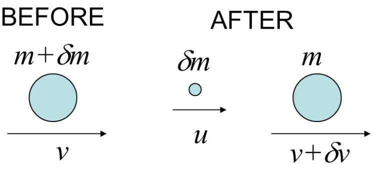
$$v^2 = v_T^2\left[1 - \exp\left(-x/x_T\right)\right]$$

3. Rocket motion

- A body of mass $m+\delta m$ has velocity v. In time δt it *ejects* mass δm, which is moving with velocity u along the line of v
- The change in mass is $m+\delta m \to m$, the change in velocity is $v \to v+\delta v$
  - Case 1: No external force

    Change of momentum:
    $$\delta p = \underbrace{m(v+\delta v)+u\delta m}_{After} - \underbrace{(m+\delta m)v}_{Before} = 0$$
  - $\delta p = mv + m\delta v + u\delta m - mv - v\delta m$
    $$= m\delta v - (v-u)\,\delta m = 0$$
  - $\delta p = m\,\delta v - \underbrace{(v-u)}_{Relative\ velocity=w}\ \delta m = 0$





- Divide by δt : $\frac{\delta p}{\delta t} = m\frac{\delta v}{\delta t} - w\frac{\delta m}{\delta t} = 0$: Let $\delta t \to 0$

  Total mass conserved $\frac{d}{dt}(m + \delta m) = 0$: $\frac{\delta m}{\delta t} \to -\frac{dm}{dt}$

  No external force $m\frac{dv}{dt} + w\frac{dm}{dt} = 0$

- Now apply an external force F

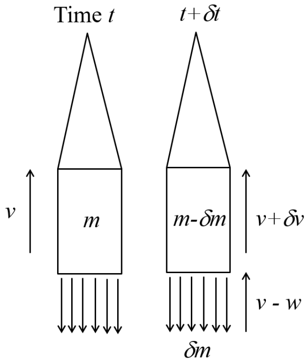
  Change of momentum $= \delta p = F\delta t = m\delta v - w\,\delta m$

  Divide by δt, let $\delta t \to 0$ and $\frac{\delta m}{\delta t} \to -\frac{dm}{dt}$

$$m\frac{dv}{dt} + w\frac{dm}{dt} = F \qquad \text{[Rocket equation]}$$

3.1 The rocket : vertical launch

- Rocket equation:
  $m\frac{dv}{dt} + w\frac{dm}{dt} = F$
- Rocket rises against gravity
  $F = -mg$
- *Eject* mass with constant relative velocity to the rocket w
- Rocket ejects mass uniformly:
  $m = m_0 - \alpha t$
  $\rightarrow \frac{dm}{dt} = -\alpha$



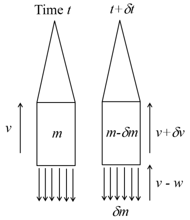
- Now consider upward motion:
- $mdv = (-mg + w\alpha)dt \quad \rightarrow \quad \int_{v_i}^{v_f} dv = \int_{t_i}^{t_f} \left(-g + \frac{w\alpha}{m_0 - \alpha t}\right) dt$
- $v_f - v_i = \left[-g(t_f - t_i) - w \log_e \frac{(m_0 - \alpha t_f)}{(m_0 - \alpha t_i)}\right]$
  $= [-g(t_f - t_i) - w \log_e (m_f / m_i)]$

Rocket vertical launch, continued

The rocket starts from rest at $t = 0$; half the mass is fuel. What is the velocity and height reached by the rocket at burn-out at time $t = T$?

- $v = \left[-gt - w \log_e \frac{(m_0 - \alpha t)}{(m_0)}\right] = \left[-gt - w \log_e \left(1 - \frac{\alpha}{m_0} t\right)\right] = \frac{dx}{dt}$
- What is the condition for the rocket to rise ? $\rightarrow \frac{dv}{dt} > 0$

  At $t = 0,\ m = m_0,\ \frac{dm}{dt} = -\alpha \ :\ \alpha w - m_0 g > 0 \ \rightarrow \ w > \frac{m_0 g}{\alpha}$



- $m = m_0 - \alpha t$; at burnout $t = T$, $m = \frac{m_0}{2} \rightarrow \alpha = \frac{m_0}{2T}$
- Maximum velocity is at the burn-out of the fuel:

  At $t = T$: $v_{max} = -gT + w \log_e 2$

  Height : $\int_0^x dx = \int_0^T \left[-gt - w \log_e \left(1 - \frac{\alpha}{m_0} t\right)\right] dt$

  - Standard integral : $\int \log_e z\, dz = z \log_e z - z$
  - $x = -\frac{gT^2}{2} + \frac{w\, m_0}{\alpha} \left[\left(1 - \frac{\alpha}{m_0} t\right)\left(\log_e \left(1 - \frac{\alpha}{m_0} t\right)\right) - \left(1 - \frac{\alpha}{m_0} t\right)\right]_0^T$
  - After simplification : $x = -\frac{gT^2}{2} + wT(1 - \log_e 2)$

4. Two-body collisions

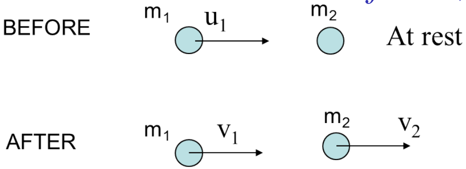
Conservation of momentum: $m_1\underline{\mathbf{u}}_1 + m_2\underline{\mathbf{u}}_2 = m_1\underline{\mathbf{v}}_1 + m_2\underline{\mathbf{v}}_2$

Conservation of energy:
$\frac{1}{2}m_1\underline{\mathbf{u}}_1^2 + \frac{1}{2}m_2\underline{\mathbf{u}}_2^2 = \frac{1}{2}m_1\underline{\mathbf{v}}_1^2 + \frac{1}{2}m_2\underline{\mathbf{v}}_2^2 \quad + \Delta E$ (= 0 if elastic)

We deal with 2 inertial frames:

- The Laboratory frame: this is the frame where measurements are actually made
- The centre of mass frame: this is the frame where the centre of mass of the system is at rest and where the total momentum of the system is zero

Elastic collisions in 1D in the Lab frame, m_2 *at rest*



Solve the conservation of energy & momentum equations in 1D

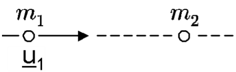
$$v_1 = \frac{m_1 - m_2}{m_1 + m_2} u_1 \quad \text{and} \quad v_2 = \frac{2m_1}{m_1 + m_2} u_1$$

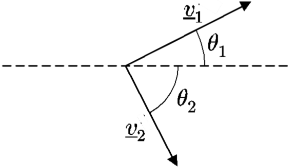
Special cases:

- $m_1 = m_2 : \;\rightarrow\; v_1 = 0,\; v_2 = u_1$
  (complete transfer of momentum)
- $m_1 >> m_2$: Gives the limits $v_1 \rightarrow u_1,\; v_2 \rightarrow 2u_1$
  (m_2 has double u_1 velocity)
- $m_1 << m_2$: Gives the limits $v_1 \rightarrow -u_1,\; v_2 \rightarrow 0$
  ("brick wall" collision)

Elastic collisions in 2D in the Lab frame: equal masses, target at rest

$$m_1 = m_2 = m\,, \quad u_2 = 0$$





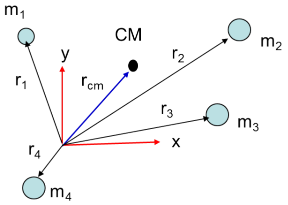
- Momentum: $m\underline{\mathbf{u}}_1 = m\underline{\mathbf{v}}_1 + m\underline{\mathbf{v}}_2 \;\rightarrow\; \underline{\mathbf{u}}_1 = \underline{\mathbf{v}}_1 + \underline{\mathbf{v}}_2$

  Squaring $\rightarrow$ $\underline{\mathbf{u}}_1^2 = \underline{\mathbf{v}}_1^2 + \underline{\mathbf{v}}_2^2 + 2\underline{\mathbf{v}}_1.\underline{\mathbf{v}}_2$
- Energy: $\frac{1}{2}m\underline{\mathbf{u}}_1^2 = \frac{1}{2}m\underline{\mathbf{v}}_1^2 + \frac{1}{2}m\underline{\mathbf{v}}_2^2 \;\rightarrow\; \underline{\mathbf{u}}_1^2 = \underline{\mathbf{v}}_1^2 + \underline{\mathbf{v}}_2^2$
- Hence $2\underline{\mathbf{v}}_1.\underline{\mathbf{v}}_2 = 0$

  $\rightarrow$ EITHER $\underline{\mathbf{v}}_1 = 0, \underline{\mathbf{v}}_2 = \underline{\mathbf{u}}_1$ OR $\theta_1 + \theta_2 = \frac{\pi}{2}$
- Either a head-on collision or opening angle is 90°

4.1 The Centre of Mass frame

- The position of the centre of mass is given by:
  $\underline{\mathbf{r}}_{\mathbf{cm}} = \frac{1}{M}\sum_{i=1}^{n} m_i \, \underline{\mathbf{r}}_{\mathbf{i}}$
  where $M = \sum_{i=1}^{n} m_i$

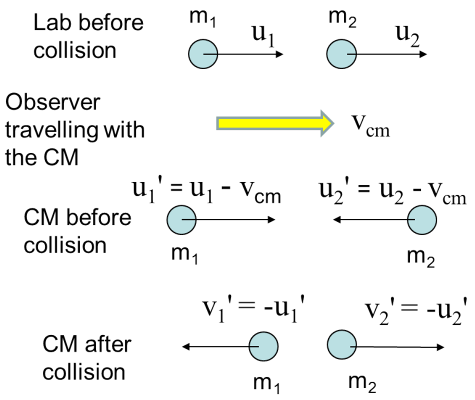


- Velocity of the CM: $\underline{\mathbf{v}}_{\mathbf{cm}} = \dot{\underline{\mathbf{r}}}_{\mathbf{cm}} = \frac{\sum_{i=1}^{n} m_i \, \dot{\underline{\mathbf{r}}}_{\mathbf{i}}}{\sum_i m_i} = \frac{\sum_{i=1}^{n} m_i \, \underline{\mathbf{v}}_{\mathbf{i}}}{\sum_i m_i}$
- Velocity of a body in the CM w.r.t. the Lab $\underline{\mathbf{v}}'_{\mathbf{i}} = \underline{\mathbf{v}}_{\mathbf{i}} - \underline{\mathbf{v}}_{\mathbf{cm}}$
- The total momentum in the CM:

$$\sum_i \underline{\mathbf{p}}'_{\mathbf{i}} = \sum_i m_i \underline{\mathbf{v}}'_{\mathbf{i}} = \sum_i m_i(\underline{\mathbf{v}}_{\mathbf{i}} - \underline{\mathbf{v}}_{\mathbf{cm}}) = \mathbf{0}$$

The total momentum of a system of particles in the CM frame is equal to zero

4.2 Two-body elastic collision in 1D : Lab to CM system

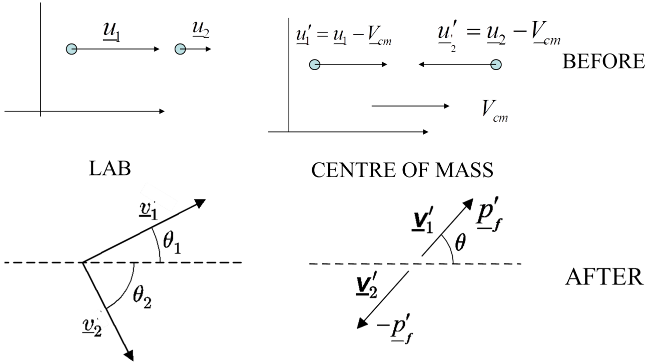


In CM, total momentum = 0, incoming and outgoing velocities are equal magnitudes and opposite direction.

- $v_{cm} = \frac{(m_1 u_1 + m_2 u_2)}{(m_1 + m_2)}$
- Before in CM :
  $m_1 u_1' + m_2 u_2' = 0$
- After in CM :
  $m_1 v_1' + m_2 v_2' = 0$
- If elastic:
  $u_1' - u_2' = v_2' - v_1'$
- Solving:
  $v_1' = -u_1'$
  $v_2' = -u_2'$

Collisions in the CM frame in 2D

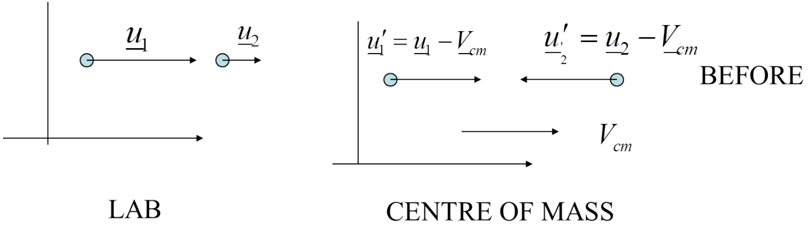
- Conservation of momentum in CM:
  $m_1 \underline{\mathbf{u}}'_1 + m_2 \underline{\mathbf{u}}'_2 = 0$; $m_1 \underline{\mathbf{v}}'_1 + m_2 \underline{\mathbf{v}}'_2 = 0$
- Conservation of energy in CM:
  $\frac{1}{2} m_1 \underline{\mathbf{u}}'^2_1 + \frac{1}{2} m_2 \underline{\mathbf{u}}'^2_2 = \frac{1}{2} m_1 \underline{\mathbf{v}}'^2_1 + \frac{1}{2} m_2 \underline{\mathbf{v}}'^2_2$



Solve the above equations :

$|v'_1| = |u'_1|$; $|v'_2| = |u'_2|$ $\rightarrow$ In CM, speeds before = speeds after

4.3 Solving collision problems in the CM frame



1) Find centre of mass velocity v_{CM}

- $(\underline{\mathbf{u}}_1 - \underline{\mathbf{v}}_{CM})m_1 + (\underline{\mathbf{u}}_2 - \underline{\mathbf{v}}_{CM})m_2 = 0$
- $\rightarrow \; \underline{\mathbf{v}}_{CM} = \frac{m_1\underline{\mathbf{u}}_1 + m_2\underline{\mathbf{u}}_2}{m_1 + m_2}$

2) Transform initial Lab velocities to CM

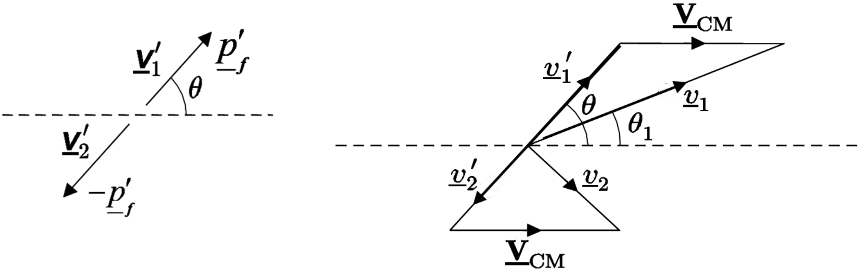
- $\underline{\mathbf{u}}_1' = \underline{\mathbf{u}}_1 - \underline{\mathbf{v}}_{CM} \quad , \quad \underline{\mathbf{u}}_2' = \underline{\mathbf{u}}_2 - \underline{\mathbf{v}}_{CM}$

3) Get final CM velocities

- $|\underline{\mathbf{v}}_1'| = |\underline{\mathbf{u}}_1'| \quad ; \quad |\underline{\mathbf{v}}_2'| = |\underline{\mathbf{u}}_2'|$

4) Transform vectors back to the Lab frame

- $\underline{\mathbf{v}}_1 = \underline{\mathbf{v}}_1' + \underline{\mathbf{v}}_{CM}$; $\underline{\mathbf{v}}_2 = \underline{\mathbf{v}}_2{}' + \underline{\mathbf{v}}_{CM}$



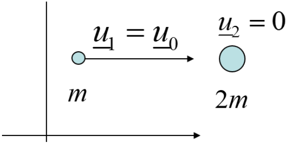
5) Can then use trigonometry to solve

Also note: $T_{Lab} = T' + \frac{1}{2} M v_{cm}^2$

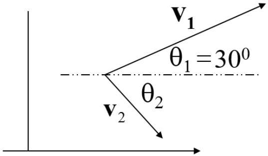
The kinetic energy in the Lab frame is equal the kinetic energy in CM + the kinetic energy of CM

Example: Elastic collision, $m_2 = 2m_1$, $\theta_1 = 30^\circ$

Find the velocities v_1 and v_2 and the angle θ_2



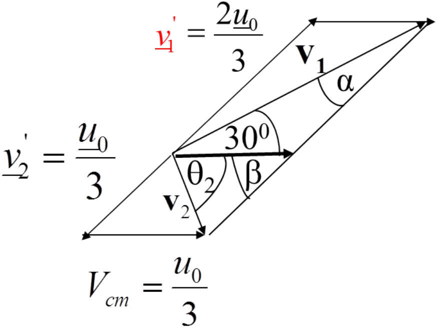
LAB before



LAB after

Magnitude of velocities:

- $v_{CM} = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} = \frac{u_0}{3}$
- $u_1' = u_0 - v_{CM} = \frac{2u_0}{3}$
- $u_2' = -v_{CM} = -\frac{u_0}{3}$
- $|v_1'| = |u_1'| = \frac{2u_0}{3}$
- $|v_2'| = |u_2'| = \frac{u_0}{3}$



Relationships between angles and speeds

- Sine rule:

  $(\sin 30 / \frac{2u_0}{3}) = (\sin \alpha / \frac{u_0}{3})$

  $\rightarrow \ \sin \alpha = \frac{1}{4} \ \rightarrow \ \alpha = 14.5^\circ$
- $\beta = 30 + \alpha = 44.5^\circ$
- $\sin 30 / \frac{2u_0}{3} = \sin(180 - 44.5)/v_1$

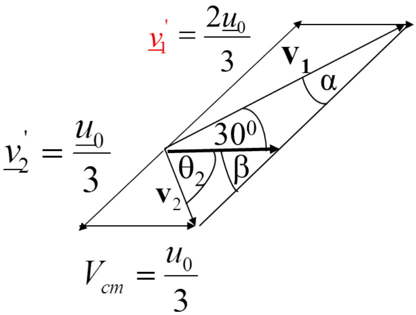
  $\rightarrow \ v_1 = 0.93 u_0$
- Cosine rule:

  $v_2^2 = (\frac{u_0}{3})^2 + (\frac{u_0}{3})^2 - 2(\frac{u_0}{3})^2 \cos \beta$

  $\rightarrow \ v_2 = 0.25 u_0$
- Sine rule:

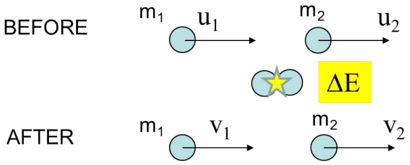
  $(\sin 44.5 / v_2) = (\sin \theta_2 / \frac{u_0}{3})$

  $\rightarrow \ \theta_2 = 68.0^\circ$



4.4 Inelastic collisions

An *inelastic* collision is where energy is lost (or there is internal excitation).



Coefficient of restitution

Defined as $$e = \frac{|\mathbf{v}_2 - \mathbf{v}_1|}{|\mathbf{u}_1 - \mathbf{u}_2|} = \frac{\text{Speed of relative separation}}{\text{Speed of relative approach}}$$

We can show $e = \sqrt{1 - \frac{\Delta E}{T'}}$ (was derived in lectures)

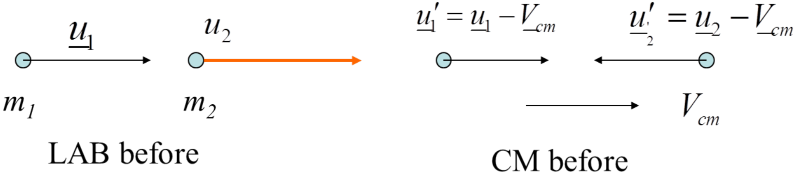
where $T' = \frac{1}{2}\mu u_1^2$ with $\mu = \frac{m_1 m_2}{m_1 + m_2}$ (the *reduced mass*)

- T' is the initial energy in the centre of mass frame, hence e is related to the fractional energy loss in this frame
- $e = 1$ completely elastic; $e = 0$ completely inelastic,

in general $0 < e < 1$

Completely inelastic collision in the CM vs. Lab

Before collision:



After collision:

- KE in CM: $T' = T_{LAB} - \frac{1}{2}(m_1 + m_2)v_{CM}^2$
- Differentiate: Loss in KE $\Delta T' = \Delta T_{LAB}$ (obvious)
- Max. energy that can be lost in the CM : $\Delta T' = T'$
- Max. energy can be lost in Lab = $\frac{1}{2}m_1 u^2 - \frac{1}{2}(m_1 + m_2)v_{CM}^2$