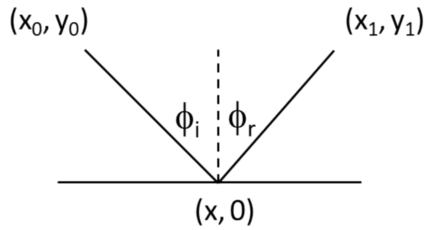
First Year Physics: Prelims CP1

Classical Mechanics: Prof. Neville Harnew

Problem Set VI Lagrangian Dynamics

Questions 1-9 are "standard" examples. Questions 10-16 are additional questions that may also be attempted or left for revision. Problems with asterisks are either more advanced than average or require extensive algebra. All topics are covered in the final 5 lectures of Hilary term.

1. Fermat's principle



A light beam is propagating in the $x - y$ plane in a media whose refraction index n depends only on y.

(a) Use Fermat's principle to show that the trajectory of the beam from (x_0, y_0) to (x_1, y_1) may be obtained by minimizing the functional

$$S(y) = c^{-1} \int_{x_0}^{x_1} n(y)[(1 + y'^2)]^{\frac{1}{2}} dx$$

where $y' = dy/dx$ and c is the speed of light in vacuum.

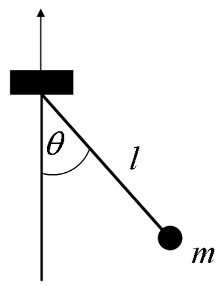
(b) Now let n be independent of x and y. A light ray propagates from (x_0, y_0) to (x_1, y_1) by reflection from the surface of a flat mirror located in the plane $y = 0$ as shown in the figure. Show that the angle of reflection ϕ_r is equal to the angle of incidence ϕ_i.

Euler-Lagrange Equation

2. Motion in two dimensions

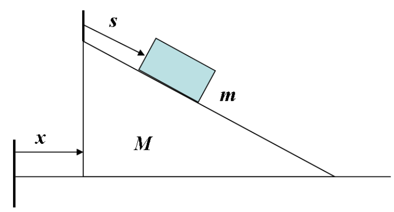
Consider a particle of mass m moving in the (x, y) plane under the influence of the potential $V(r)$ where $\mathbf{r}$ is the postion vector of the particle in an inertial reference frame.
Construct the Lagrangian and the Hamiltonian of the particle in polar coordinates r, θ, hence find which quantities are constants of motion.

3. The simple pendulum



(a) Use the E-L equation to calculate the period of oscillation of a simple pendulum of length l and bob mass m in the small angle approximation.
(b) Now assume that the pendulum support is accelerated in the vertical direction at a rate a, find the period of oscillation. For what value of a the pendulum does not oscillate? Comment on this result.

4. A sliding block

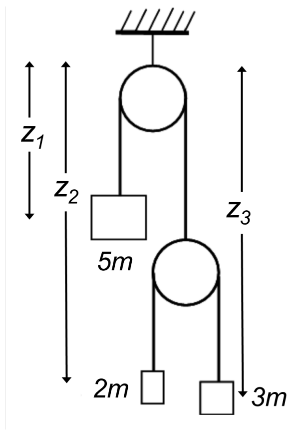


A block of mass m slides on a frictionless inclined plane of mass M, which itself rests on a horizontal frictionless surface.

(a) Choose the displacement of the inclined plane x and the displacement of the block relative to the inclined plane s as generalized coordinates and find the Lagrangian of the system.

(b) Write down the E-L equation for each coordinate and find the acceleration of the inclined plane.

5. Atwood's machine



The three masses shown move in a vertical plane under the influence of constant gravity and the tension in the unextendable strings. Assuming that the pulleys are massless and that all friction forces can be neglected:

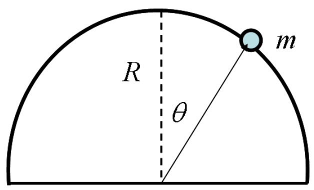
(a) Write down the constraints equation in terms of z_1, z_2 and z_3 that result from the fixed length of the strings, each length L, hence show that the motion of the three masses may be described by two generalized coordinates.

(b) Use the E-L equation to find the acceleration of each mass.

(c) Find the tensions in the two strings; hence show the tension in the upper string is twice that of the lower.

Why does the upper pulley rotate despite the fact that the masses on either side are equal?

6. Particle sliding on a sphere



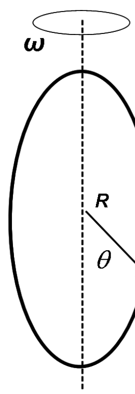
A particle of mass m slides without friction down the surface of a hemisphere of radius R.

(a) Construct the Lagrangian of the problem in terms of the polar coordinates (r, ϑ), in the range when the constraint $r = R$ is valid. Find the equation of motion.

(b) The drawback of imposing the $r = R$ constraint in part (a) is that it does not allow the normal force (i.e. that the hemisphere applies to the particle) to be calculated. To find this force we must introduce the radius of the particle r as an extra parameter (but which only varies by an infinitesimal amount) i.e. rewrite the Lagrangian now with r as a free variable. We must also include in the Lagrangian a potential term $V(r)$ due to the reaction force causing an infinitessimal deformation of the hemisphere. Write down this new Lagrangian and then find the reaction force $F_N = -\frac{\partial V}{\partial r}$ when $r = R$ and $\dot{r} = \ddot{r} = 0$.

(c) Assuming that the particle is released from the top of the sphere from rest, show that the particle leaves the surface at an angle $\cos\vartheta_{max} = 2/3$.

7. A bead on a rotating hoop



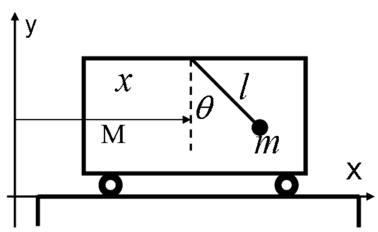
A vertical circular hoop of radius R rotates about a vertical axis at an angular velocity ω. A bead of mass m can slide on the hoop without friction and is constrained to stay on the hoop. By taking the angle ϑ between the radius line and the vertical, as a generalized coordinate:

(a) Find the Lagrangian and the equation of motion in the ϑ coordinate.

(b) Show that there are three equilibrium positions of the bead. Discuss the stabiliy of each equiibrium point and find the frequency of small oscillations about the stable ones.

(c) Find the the total energy $T + V$ and the Hamiltonian of the system. Demonstrate that the Hamiltonian is a constant of the motion but the total energy is not. Why is this?

8. A pendulum with accelerated support



A box of mass M can slide horizontally on a frictionless surface. A simple pendulum of string length l and mass m, is suspended from the ceiling of the box above its centre of mass. Denote the coordinate of the centre of mass of the box by x and the angle that the pendulum makes with the vertical by θ. At $t = 0$ the pendulum displacement is $\theta = \theta_0 \neq 0$

(a) Find the Lagrangian and the equations of motion for the generalized coordinates x and θ.

(b) Find the solutions for x and θ in the small angle approximation, hence show that both the pendulum and the box execute SHO about their centre of mass at an angular frequency

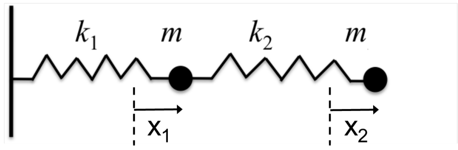
$$\omega = \left[\frac{M+m}{M}\right]^{\frac{1}{2}} \left(\frac{g}{l}\right)^{\frac{1}{2}} .$$

9. Normal modes

Two equal masses m are connected by two massless springs of force constants k_1 and k_2 as shown, and are free to move in the x direction. The system is placed on a horizontal frictionless table and attached to the wall.

(a) Write the Lagrangian of the system using the coordinates x_1 and x_2 that give the displacements of the masses from their equilibrium positions. Use the E-L equations to find the equation of motion of each mass.

(b) Find the solutions in which the two masses execute simple harmonic oscillations at the same frequency (normal modes). What are the common angular frequencies of oscillation for the special cases $k_1 \ll k_2$ and $k_1 \gg k_2$.



Additional questions

10. Revisit Q6 of problem sheet 5.
A compound pendulum consists of a uniform circular disk of radius a and mass m with a series of small holes drilled at regular intervals along a diameter. A horizontal axis about which the disk is free to turn is placed through the hole O', a distance x from the centre of the disk (O).

Find the period T of small-angle oscillations using the Euler-Lagrange equation.

11. A small uniform cylinder of radius a rolls without slipping on the inside of a large, fixed cylinder of radius $b\,(b \geq a)$. Use the E-L equation to show that the period of small oscillations of the rolling cylinder is that of a simple pendulum of length $3(b-a)/2$.

12. Revisit Q3 of problem sheet 5.
A cylindrical reel of thread of radius r and mass m is allowed to unwind under gravity, the upper end of the thread being fixed. (a) Find the initial acceleration of the reel using the E-L equation (b) the initial tension in the thread. [The moment of inertia of a uniform cylinder of radius r and mass m about its axis is $\frac{1}{2}mr^2$.]

[Ans: (a) $2g/3$ m s^{-2}, (b) $mg/3$ N]

13. A ladder of length l and mass m stands on a frictionless floor and leans on a frictionless wall. The ladder is allowed to slide while remaining supported by the floor and wall, and its inclination relative to the vertical wall is given by the angle θ.

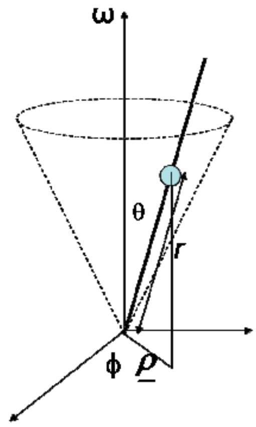
(a) As the ladder slides, show that the centre of mass (CM) of the ladder moves in a circular path, the centre being the point where the floor meets the wall.

(b) Write down the kinetic and potential energy of the ladder. [Hint: you can separate the KE into rotational energy about the CM and transational energy of the CM.] Assume the moment of inertia of the ladder is $ml^2/12$.

(c) Show that the constraints allow you to write the Lagrangian in a single coordinate θ. Determine the equation of motion using the E-L equation.

(d) Use the Lagrangian to discuss the conservation of the total energy E, the Hamiltonian H and the angular momentum J.

14.* Rotating bead



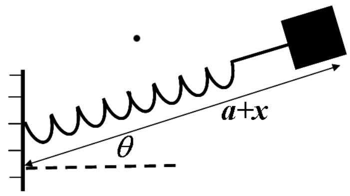
A bead of mass m is constrained to slide on a frictionless wire which is made to rotate about a vertical axis at an angular velocity ω. The wire is tilted away from the vertical by an angle θ and the location of the bead is measured by the coordinate r.

(a) Write down the equation of motion of the bead using the E-L equation. Test the integrity of your equation by taking extreme values of θ.

(b) Find the general solution assuming that at $t = 0, r = r_0, \dot{r} = 0$. Based on this solution, show that for $r_0 = g\cos\theta/\omega^2\sin^2\theta$, the bead moves in circular motion. Describe the motion for $r < r_0$ and $r > r_0$.

(c) Which of the following quantities is a constant of the bead motion: angular momentum with respect to the origin, the Hamiltonian, total energy?

15. * 2-D spring



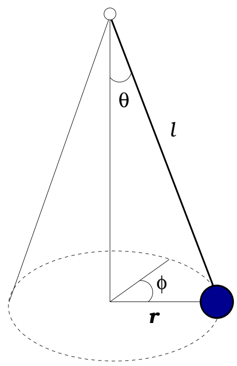
Revisit Q12 of problem sheet 4.
A particle of mass m is attached to the free end of a massless spring of equilibrium length a and spring constant k. The other end of the spring is pivoted to a frictionless horizontal surface and the particle is allowed to move in 2-D in the horizontal plane under the influence of the spring force which is assumed to obey Hook's law.

(a) Write the E-L equations for the polar coordinate (r, θ). Identify the cyclic coordinates and the corresponding conserved quantities. Determine the equation of motion in terms of the variable r.

(b) Write the total energy of the system (for a given angular momentum J) and, using the concept of effective potential, find the radius for circular orbit. Show that it is consistent with the value obtained from Newton's second law.

(c) It was shown in Q12 problem set 4 that if the rest length of the spring is negligible ($a \approx 0$), the path of the particle is elliptical, with r measured from the centre of the ellipse. Use the expression for total energy E to find the major and the minor axes of the ellipse.
[Answer: $a, b = r_0(\sqrt{1 \pm \epsilon})$ where $r_0 = E/k$ and $\epsilon = \sqrt{1 - kJ^2/mE^2})$]

16. * The Spherical Pendulum



Consider a spherical pendulum which consists of a mass m suspended by a massless string of length l from the ceiling as show below.

(a) Write the Lagrangian of the system in terms of the polar coordinates (θ, ϕ).

(b) Show that ϕ is a cyclic coordinate and find the corresponding conserved quantity. Find the equation of motion in θ and hence show that the pendulum can move steadily around a circle with $\theta = \theta_0$ at an angular frequency Ω given by:

$$\Omega^2 = \frac{g}{l\cos\theta_0}.$$

(c) By relaxing the constraint on l in analogy to Question 6, find the tension in the string. Show that the expression for the tension reverts to that expected from Newton II for circular motion.