First Year Physics: Prelims CP1

Classical Mechanics: Prof. Neville Harnew

Problem Set IV Central Forces and Orbits

Questions 1-11 are "standard" examples. Questions 12-15 are additional questions that may also be attempted or left for revision. Problems with asterisks are either more advanced than average or require extensive algebra. All topics are covered in lectures 6-10 of Hilary term.

1. General circular motion: A particle of mass m is constrained to slide on the inside of a vertical smooth semi- circular ring of radius r. The position of the particle is described by a polar coordinate system whose origin is at the centre of the circle with axes along the orthogonal unit vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ where θ is the angle between the radius vector $\boldsymbol{r}$ and the the vertical line that passes through the origin.
(a) Derive the general relationships for velocity and acceleration in polar coordinates. Hence show that the acceleration of the particle may be written as

$$\boldsymbol{a} = r\ddot{\theta}\hat{\boldsymbol{\theta}} - \frac{v^2}{r}\hat{\boldsymbol{r}}$$

where $v = r\dot{\theta}$ is the magnitude of the particle velocity.
(b) Use the equation of motion in the $\hat{\boldsymbol{\theta}}$ direction to calculate the period of small oscillation about the bottom point.
(c) Use the equation of motion in the $\hat{\mathbf{r}}$ direction to get an expression for the reaction force exerted by the surface as a function of θ. Assuming the particle is released from rest at the top of the semi-circle, use this expression and the conservation of energy to show that the magnitude of the reaction force at $\theta = 60^\circ$ is $\frac{3}{2}mg$.

2. Motion under a central force: A particle is moving under an attractive force $f(r)$ per unit mass directed along the line joining the particle to a fixed point O. The vector displacement of the particle from O at time t is $\mathbf{r}$. Show that $\mathbf{r} \times \dot{\mathbf{r}}$ is a constant vector and hence that the particle moves in a plane. Using plane polar coordinates (r, θ) for motion within the plane: define the angular momentum J; using the transverse equation of motion show that $|J|$ is a constant; show that the radius vector r sweeps out area at a constant rate.

3. Motion on 2-D surface: Two particles of mass m are connected by a light inextensible string of length l. One of the particles moves on a smooth horizontal table in which there is a small hole. The string passes through the hole so that the second particle hangs vertically below the hole.
(a) Use the conservation of energy and angular momentum or otherwise to show that:

$$\dot{r}^2 \ = \ A - B/r^2 - gr$$

where $r(t)$ is the distance of the first particle from the hole, A and B are constants and g is the acceleration due to gravity. Initially the particle on the table is at a distance $l/2$ from the hole and moves with a speed v_0 directed perpendicular to the string. Find the values of A and B.
(b) Show that the condition for the particle on the table to move in circular motion is given by $v_0^2/gl = \frac{1}{2}$.
[Ans: $A = (gl + v_0^2)/2$, $B = (lv_0)^2/8$.]

4. Simple orbits

(a) If a communication satellite is to remain in orbit constantly above a particular city on the equator, what distance above this city will the orbit be? Why must the city be on the equator?
(b) A binary star consists of two stars bound together by gravity. From spectroscopic studies of Plaskett's binary it is known that the period of revolution of the stars about their centre of mass is 14.4 days, the speed of each component is $220\,\mathrm{km\,s^{-1}}$ and they are moving roughly in opposite directions along a nearly circular orbit.
Argue that the masses of the two component stars must be roughly equal. Assuming a circular orbit, find the distance between the two stars and their masses.

[Ans: (a) 35886 km; (b) 8.7×10^{10} m, 1.25×10^{32} kg,]

5. Rotation of galaxies: A galaxy may be modelled as a dilute spherical distribution of stars with constant density within a radius R and total mass M. Find expressions for the velocity v of an individual star of mass m in a circular orbit about the centre of the galaxy at distances r both greater than and less than R. [For the latter calculation you will need to use the equivalent of Gauss Law: the gravitational force at an interior point r is given by the mass contained within a sphere of radius r.] Sketch the dependence of v on r. Observations of $v(r)$ show that for many galaxies the measured curve requires a larger mass than that determined from the luminous matter. This is the so-called *dark matter* problem and it is a hot topic in current astrophysical research.

6. Putting satellite into orbit: It is required to put a satellite into an orbit with apogee (furthest distance from the centre of the planet) of $5R/2$, where R is the radius of the planet. The satellite is to be launched from the surface with a speed v_0 at 30° to the local vertical. If M is the mass of the planet, use the conservation of energy and angular momentum to show that $v_0^2 = 5GM/4R$. Assume that the planet is not rotating and that effects due to the planetary atmosphere can be ignored. (Adapted from French & Ebison).

7. Deflection by a central force

An incoming comet of mass m is detected at the edge of the solar system. It has a velocity v and an impact parameter b (the minimum distance between the centre of the sun and the extrapolation of the comet's initial path) with respect to the Sun (mass M).
(a) Sketch the expected trajectory of the comet. Indicate the impact parameter b and the angle of scattering (deflection) ϕ on your plot.
(b) Show that the total energy of the comet as a function of position vector $\mathbf{r}$ from the centre of the Sun may be written as:

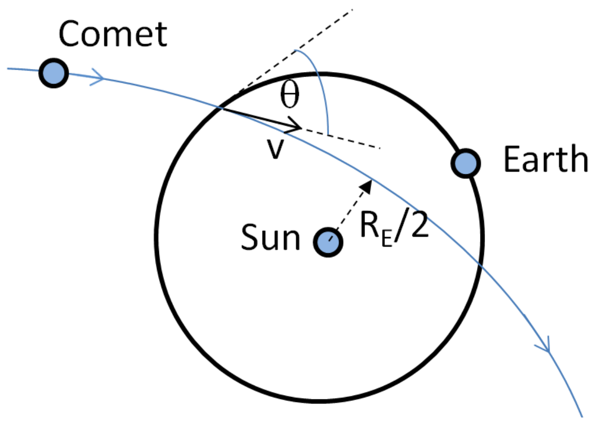
$$E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{\alpha}{r}\,.$$

Find expressions for J and α.
(c) Using the result in (b) or otherwise, show that the distance of closest approach to the centre of the Sun is

$$r_{min} = \frac{GM}{v^2}\left[\sqrt{1+\left(\frac{bv^2}{GM}\right)^2} - 1\right]$$

where G is the gravitational constant.

8. The comet of Question 7 is deflected by the Sun and on its trajectory crosses the Earth's orbit. The distance of closest approach to the Sun is measured to be half the distance between the Earth and the Sun, R_E, assuming that the Earth's orbit is circular. The velocity at that point is 60 kms^{-1}, which happens to be twice the orbital speed of the earth. What is the comet's speed v when it crosses the Earth's orbit? What angle θ does the comets trajectory make with the Earth's orbit at that point? [You may neglect the gravitational attraction between the comet and the Earth.]



9. Changing orbit: A satellite of mass m is put into a circular orbit of radius r_0 around the Earth (of mass M).

(a) Given that the speed of the satellite is v_0, show that

$$r_0 = \frac{GM}{v_0^2},$$

where G is the gravitational constant.

(b) Given that J_0 is the satellite's angular momentum and E_0 its total energy (kinetic plus potential), show that

$$J_0 = \frac{GMm}{v_0}\,, \quad \text{and} \qquad \mathrm{E}_0 = -\frac{1}{2}\mathrm{mv}_0^2$$

(c) Whilst in its circular orbit, the satellite is given an instantaneous impulse which changes its angular momentum from J_0 to αJ_0, where $\alpha < 1$. The satellite's energy remains unchanged. Describe the shape of the new orbit and, using the result of Question 7(b), calculate the minimum and maximum values for the distance between the satellite and the centre of the Earth if $\alpha = 12/13$.

(d) Assuming that whilst the satellite in its circular orbit the impulse leaves the angular momentum unchanged but increases its kinetic energy by 1%. Find the difference between the maximum and minimum distances from the centre of the earth.

Sketch the orbits of (c) and (d) comparing with the original circular orbit.

[Ans: (c) $r_{max}/r_0 = 1 + (1-\alpha^2)^{1/2}$, $r_{min}/r_0 = 1-(1-\alpha^2)^{1/2}$ (d) $0.20r_0$]

10. Rutherford scattering: A particle of electric charge q_1 and mass m is projected, with velocity v_0 and impact parameter d ($d > 0$), towards a fixed nucleus of opposite charge q_2 (attractive force). Sketch the trajectory. Show that the angle of deflection α is given by:

$$\cot\left(\frac{\alpha}{2}\right) \;=\; \frac{4\pi\epsilon_0}{q_1 q_2} d v_0^2 m\;.$$

By making the replacement $\dfrac{q_1 q_2}{4\pi\epsilon_0} \to GMm$, notice that the above result may be adapted to find the angle through which the path of a comet of mass m is ultimately deflected as it passes the Sun.

11. Two-body dynamics

Consider an isolated two-body system of masses m_1 and m_2 moving under their internal gravitational interaction. The particles can be treated as point masses and their position in an inertial frame is described by the vectors $\mathbf{r_1}$ and $\mathbf{r_2}$.

(a) Show that the equation of motion for either particle may be written as

$$-\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}} = \mu\ddot{\mathbf{r}}$$

where $\mathbf{r} = \mathbf{r_1} - \mathbf{r_2}$ is the relative position vector and $\mu = \frac{m_1m_2}{m_1+m_2}$ is the reduced mass.

(b) Show that if the motion is viewed in the centre of mass frame, the kinetic energy of the two particles is given by $T = (1/2)\mu\dot{\mathbf{r}}^2$.

(c) Show that if the motion is viewed in the centre of mass frame, the total angular momentum of the system is given by $\mathbf{J} = \mu\mathbf{r} \times \dot{\mathbf{r}}$.

Conclude therefore that the motion of the two particles may be analysed using an equivalent system comprising a *single particle* of mass μ moving under the influence of an attractive central force $f(r)\hat{\mathbf{r}}$ where $\mathbf{r}$ is the position vector of the particle with respect to the other.

Additional questions

12. Elliptical orbit with a spring-like force

A particle of mass m moves in the $(x.y)$ plane under the influence of a spring-like force of the form $\mathbf{F} = -k\mathbf{r}$, where $k > 0$ is a positive constant. Assuming that at $t = 0$ the position of the particle is given by $\mathbf{r}_0 = (x_0, 0)$ and the velocity by $\mathbf{v}_0 = (0, u_0)$,

(a) Write the EOM in Cartesian coordinates and show that the path of motion is an ellipse centred at the origin of coordinates. Find the major and the minor axes of the ellipse.

(b) Derive an expression for the effective potential and use it to analyse the motion and to show that it is always bound.

13.* Motion of a binary star

The two individual stars in a binary system $(m_1 = m_0, m_2 = 2m_0)$ are in circular orbit about their common centre of mass and are separated by a distance r_0. At some stage, the more massive star explodes and as a result suffers a spherically symmetric loss of mass. After the explosion the masses of the starts become equal:

(a) Using circular motion dynamics, calculate the total energy and period of the binary star as it is viewed before the explosion in the centre of mass frame.

(b) Now calculate the total energy and period of the equivalent single-particle orbit system before the explosion.

(c) Calculate the total energy of the binary star after the explosion in its new centre of mass frome. Hence show that the binary star will remain intact (bound) after the explosion.

(d) Sketch the equivalent single particle orbit and the actual binary orbits as viewed in the new centre of mass frame after the explosion.

[Ans: (a) $E = -\frac{Gm_0^2}{r_0}, T = 2\pi\sqrt{\frac{r_0^3}{3Gm_0}}$: (b) Same result as (a) : (c) $E = -\frac{1}{4}\frac{Gm_0^2}{r_0}$]

14. An efficient way to reach the Moon is to first put the spacecraft in a low circular Earth orbit (radius r_0, speed v_0). The speed is then increased to v_p via a boost of the rocket, giving an elliptical orbit with apogee r_a at the Moon's orbital radius about the Earth, and perigee at r_0. Show that

$$\left(\frac{v_p}{v_0}\right)^2 = \frac{2r_a}{r_0 + r_a}.$$

Taking $r_0 \approx R_e$, $r_a \approx 60R_e$, calculate v_0 and v_p.

Calculate the fractional error on the spacecraft's apogee with respect to r_0 if v_p/v_0 is incorrect by a fraction f. Hence show that a 0.1% error in the boosted velocity gives a greater than 10% error in the spacecraft's apogee at the Moon.

What other considerations are necessary for a successful rendezvous between spacecraft and Moon? [$R_e = 6.38 \times 10^6$ m] (Adapted from Fowles & Cassidy.)

15. Consider a planet orbiting the sun in an elliptic orbit. Let r and r' be the distance from the planet to the two foci. An alternative and equivalent definition to the standard cartesian equation of an elipse is that $r + r' = \text{constant} = 2a$, where a is the length of the major axis. The separation between the two foci of the ellipse is $2a\epsilon$ where ϵ is the eccentricity. Let the angle between the line joining the two foci and a line from one focus to the planet be θ.

(a) Show that

$$\frac{1}{r} = \frac{1 - \epsilon\cos\theta}{a(1-\epsilon^2)}$$

(b) Show that $r^2\dot{\theta} = C$ where C is a constant. Hence using the result in part (a) show that

$$\ddot{r} = -\frac{\epsilon\, C}{a(1-\epsilon^2)}\cos\theta\,\dot{\theta}$$

(c) Using the result in (b) show that the acceleration in the radial direction

$$a_r = -\frac{k}{r^2}$$

and find the constant k. Discuss the signifiance of this result for the law of gravity.