First Year Physics: Prelims CP1

Classical Mechanics: Prof. Neville Harnew

Problems I: Introductory problems & collisions in one dimension

Questions 1-9 are standard examples. Questions 10-12 are additional questions that may also be attempted or left for revision. Problems with asterisks are either more advanced than average or may require significant algebra. All topics are covered in lectures 1-5 of Michaelmas term.

Introductory problems

1. Vectors in two dimensions: The position vector of a particle moving in the x-y plane is given by a vector of magnitude r and an angle θ with the x-axis. At $t = 0, \mathbf{r} = 4\mathbf{i} + 3\mathbf{j}$ (cm).

(a) Find a unit vector $\hat{\mathbf{u}}$ that makes an angle $\theta = 20°$ with $\mathbf{r}$.

(b) Show by geometrical construction that any vector in the x-y plane may be obtained by linear combination of $\mathbf{r}$ and $\hat{\mathbf{u}}$.

(c) Assume that at t=0 the particle starts to move in circular motion about the origin, at a *constant* speed $v = 15\ \mathrm{cm\,s^{-1}}$. Explain why the motion is still *accelerated.* Show by geometrical illustration that the velocity is tangent to the path.

(d) Find the dependence of θ on time, hence obtain expressions for $\mathbf{r}$ and the velocity $\mathbf{v}$ as a function of time.
[Ans: (a) $\hat{\mathbf{u}} = 0.55\mathbf{i} + 0.84\mathbf{j}$ (or $\hat{\mathbf{u}} = 0.96\mathbf{i} + 0.29\mathbf{j}$), (d) $\theta = 3t + 0.64$ (rad)]

2. Dimensional analysis

(a) Consider the period of a simple pendulum with length ℓ and bob of mass m moving under gravity. Show on dimensional grounds that a likely relationship for the period is

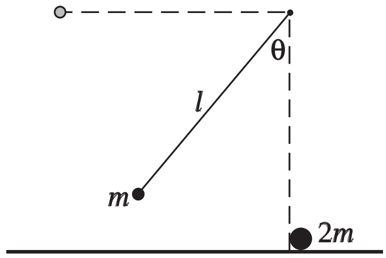
$$T = kf(\theta_0)\sqrt{\ell/g}, \quad \text{where } \theta_0 \text{ is the angular amplitude,}$$

k a dimensionless constant and f an arbitrary function.

(b) For a satellite with mass m, in a circular orbit of radius R about a planet of mass M ($M \gg m$), use the method of dimensions to show that the period of revolution is given by $T^2 = k\dfrac{R^3}{GM}$, where G is Newton's constant.

(c) Use Coulomb's law for the force between two charges q_1, q_2 a distance r apart, $F_E = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_1 q_2}{r^2}$, to find the dimensions of ϵ_0. Next, use the expression for the force between two parallel wires of length ℓ, a distance d apart, carrying currents I_1, I_2, $F_M = \dfrac{\mu_0}{4\pi}\dfrac{I_1 I_2}{d}\ell$, to find the dimensions of μ_0. Finally, show that the combination $\sqrt{\dfrac{1}{\mu_0\epsilon_0}}$ has dimensions [$\mathrm{LT^{-1}}$] and find its numerical value from a table of physical constants. Comment on the result.

3. Energy conservation: A pendulum bob of mass m, at the end of a string of length ℓ, starts from rest with the string taut and horizontal – see figure.



At the lowest point of swing the bob strikes a stationary object of mass $2m$ resting on a frictionless horizontal surface. The collision is elastic. Find (using energy considerations):

(a) the speed of the bob just before impact,

(b) the tension in the string at the same instant,

(c) the velocity given to the object by the impact.

(d) After the impact the bob will recoil to reach a new maximum height before resuming oscillations. Find the maximum angle (to the vertical) that the string makes in subsequent oscillations of the bob.

[Ans: (a) $\sqrt{2g\ell}$, (b) $3mg$, (c) $2\sqrt{2g\ell}/3$, (d) 27.3°.]

4. The Simple harmonic oscillator: A particle of mass m constrained to move in the x-direction only is subject to a force $F(x) = -kx$, where k is a constant. Show that the equation of motion can be written in the form

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0, \quad \text{where} \quad \omega_0^2 = k/m \, .$$

(a) Show by direct substitution that the expression

$$x = A\cos\omega_0 t + B\sin\omega_0 t$$

where A and B are constants, is a solution and explain the physical significance of the quantity ω_0. Show that an alternative solution may be expressed as

$$x = x_{\text{max}}\cos(\omega_0 t + \phi)$$

where x_{max} is the amplitude and ϕ is the phase constant of oscillation.

(b) Find in terms of m and ω_0 the change in the potential energy $U(x) - U(0)$ of the particle as it moves from the origin. Explain the physical significance of the sign in your result.

(c) The potential energy is subject to an arbitrary additive constant; it is convenient to take $U(0) = 0$. The kinetic energy $T(x) = \frac{1}{2}mv^2$. Show by differentiation that the particle's total energy ($E = T+V$) is constant. Express E as a function of the particle's (a) maximum displacement x_{max} and (b) maximum velocity v_{max}.

5. The potential energy function: A particle of mass m, starting from $x = -\infty$, approaches a force region whose potential energy is given by

$$U(x) = \frac{U_0\, a^2}{x^2 + a^2} \, ,$$

where $U_0 > 0$ and a are constants.

(a) Derive an expression for the force on the particle as a function of x.

(b) Draw rough graphs of the force and the potential as functions of x.

(c) State in which region the force is attractive and which repulsive, and explain how your answers can be understood in terms of your graphs.

(d) If a particle is released at rest from the origin, with what velocity will it be travelling at very large distances?

(e) What is the least velocity it must be given at $x = -\infty$ which will allow it to reach $+\infty$?

6. A two-particle problem in 1-D - the centre of mass system: Consider two particles m_1 and m_2 moving in one dimension under the influence of an attractive force of magnitude F_{int} between them. Each is also acted on by an external force: F_1 and F_2, respectively. Let the particles have co-ordinates x_1, x_2 respectively.

(a) Write down the equations of motion of the two masses separately. By adding your two expressions, get a single equation of motion for the system in terms of F_1 and F_2.

(b) Hence show that if there are no external forces present (i.e. $F_1 = F_2 = 0$) the momentum of the system is conserved.

(c) Write down an expression for the position $X_{\rm cm}$ of the centre of mass (CM) of the system. By differentiating your expression, show that the momentum of the system is the same as that of a mass $M = m_1 + m_2$ moving with the velocity of the centre of mass.

(d) Differentiate again, and compare your result with the equation obtained in (a) above. Hence show that the acceleration of the centre of mass is as if all the mass were concentrated there and the resultant external force acted through it. What happens to the centre of mass of the system if there are no external forces?

(e) Assume that the two masses are isolated in an inertial reference frame (no external forces: $F_1 = F_2 = 0$) and consider the motion of the particles in the CM frame using the coordinates $x'_i = x_i - X_{\rm cm}$ $(i = 1, 2)$. Show that the equation of motion of the system in the CM frame under the influence of the internal force is given by Newton's second law as $\mu\ddot{x}' = F_{int}$ where μ is the *reduced mass* of the two particle system, $1/\mu = 1/m_1 + 1/m_2$ and $x' = x'_1 - x'_2$ is the relative distance between the two particles.

Collisions in one dimension

7. Elastic equal mass collision: A runaway (assume frictionless) railway truck A of mass m is moving along the track at velocity $u_\circ$ and hits a stationary truck B of identical mass m. By considering the conservation of linear momentum and energy, assuming the collision between the trucks to be *elastic*, determine the velocities, v_A and v_B, of the two trucks after the collision.

Now consider the same collision as viewed by an observer O' located at the *Centre of Mass* (CM) and moving at constant velocity V_{CM} with respect to B.

(a) Calculate V_{CM} and explain why it remains unchanged after the collision.

(b) Show that the total momentum in the CM frame is zero (hence the alternative name *zero momentum frame*).

(c) Determine the initial and final velocities of the two trucks as viewed by O', and illustrate your solution both before and after the collision with simple, clearly labelled sketches. Notice that in the CM frame, the magnitude of the velocity of each truck remains unchanged after the collision.

(d) Discuss the advantages of observing collisions in the CM frame.

8. Elastic unequal mass collision: We have a similar situation as in the previous question but now truck A has mass m_A and initial velocity u_A. Truck B has mass m_B and is initially at rest. The collision is elastic.

(a) Show the subsequent velocities v_A, v_B in the laboratory frame are:
$v_A = (m_A - m_B)u_A/(m_A + m_B)$, $\quad v_B = 2m_A u_A/(m_A + m_B)$.
On the basis of these results, find the velocities in the two limiting cases:
(i) $m_A >> m_B$; $\quad$ (ii) $m_A << m_B$.

(b) Find the subsequent initial and final velocities u'_A, u'_B and v'_A, v'_B in the CM Frame.

9. Inelastic equal mass collision: Consider the problem of the colliding equal-mass railway trucks in problem 7. Suppose now that the collision is *inelastic*, such that half the initial kinetic energy is 'lost' during the collision.

(a) By considering the conservation of linear momentum and the changes in the total kinetic energy of the system, determine the velocities, v_A and v_B, of the two trucks after the collision in the laboratory frame. Discuss where the 'lost' kinetic energy might have gone (in what sense is it really 'lost'?).

(b) Newton's coefficient of restitution e in a 2-body collision is defined as the ratio of the magnitude of the relative velocity of separation to the magnitude of the relative velocity of initial approach:

$$e = \frac{|\mathbf{v_B} - \mathbf{v_A}|}{|\mathbf{u_B} - \mathbf{u_A}|}.$$

Determine the value of e for the inelastic collision in (a) above.

(c) Consider the same inelastic collision in (a) as viewed by the an observer in the CM frame. What are the final velocities v'_A and v'_B of the two trucks in this frame. Verify that the value of the coefficient of restitution e' in this frame is the same as in part (b).

Additional questions

10. Energy loss to rest: A highly (though not perfectly) elastic ball is released from rest at a height h above the ground and bounces up and down. With each bounce a fraction f of its kinetic energy just before impact is lost. Hence show that the height reached on the nth bounce is $h_n = (1-f)^n h$. Find the time taken for the ball to drop from height h_n and subsequently reach h_{n+1}. Hence find an expression for the time taken for the ball to come to rest and evaluate it for the case when h is 5 m and f is 0.1.

[Ans: time to rest is $\sqrt{2h/g}(1+\epsilon)/((1-\epsilon)$ where $\epsilon = \sqrt{1-f}$; 38.3s.]

11. * Force and momentum - falling chain:

(a) A uniform chain of length L and mass m is stretched out on a frictionless horizontal table with part of its length h hanging down through a hole in the table. Assuming that the chain is released from rest, how long will it take the chain to fall off. Neglect the friction between the hole and the chain.

(b) The chain in (a) is held stretched vertically just above the surface of a weighing scale and then released from rest. What is the reading of the scale when half of the length of the chain has fallen down?

[Ans: (a)$\sqrt{L/g}\cosh^{-1}(L/h)$ (b) $\frac{3}{2}mg$]

12.* The equation of motion: constant force-sliding blocks: A block of mass m slides on the frictionless surface of an inclined plane of angle θ, which itself has a mass M. The plane slides on a horizontal table. Assuming no friction between the surfaces:

(a) Write the equations of motion of the block and the inclined plane in vectorial form as viewed in the inertial reference frame of the table.

(b) By resolving the equations in (a) into components parallel and perpendicular to the surface of the inclined plane, or otherwise, find the acceleration of the block along the plane and also that of the inclined plane along the table.

(c) Calculate the internal force that the block and the inclined plane apply on each other.

(d) Show that the horizontal component of the linear momentum of the total system, ie. the plane plus block, remains constant.

[Ans: (b) $g\frac{\sin\theta(1+\frac{M}{m})}{\sin^2\theta+\frac{M}{m}}$, $-g\frac{\sin\theta\cos\theta}{\sin^2\theta+\frac{M}{m}}$, (c)$Mg\frac{\cos\theta}{\sin^2\theta+\frac{M}{m}}$]