LECTURES 20 - 29

INTRODUCTION TO CLASSICAL MECHANICS

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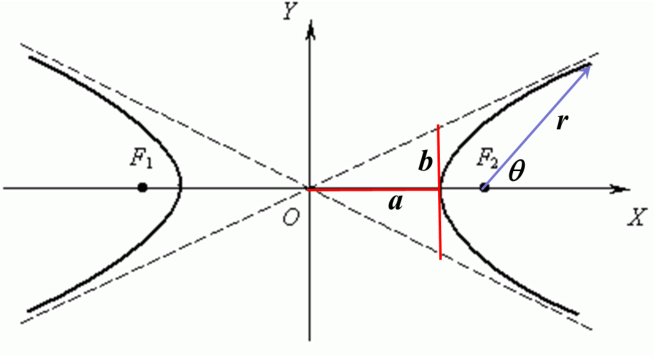
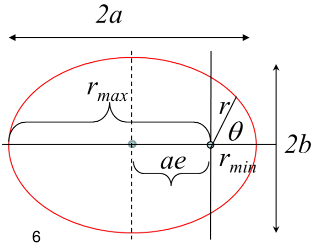
OUTLINE : INTRODUCTION TO MECHANICS LECTURES 20-29

29.2 Normal modes of coupled identical springs

29.3 Final example: a rotating coordinate system

20.1 The hyperbolic orbit



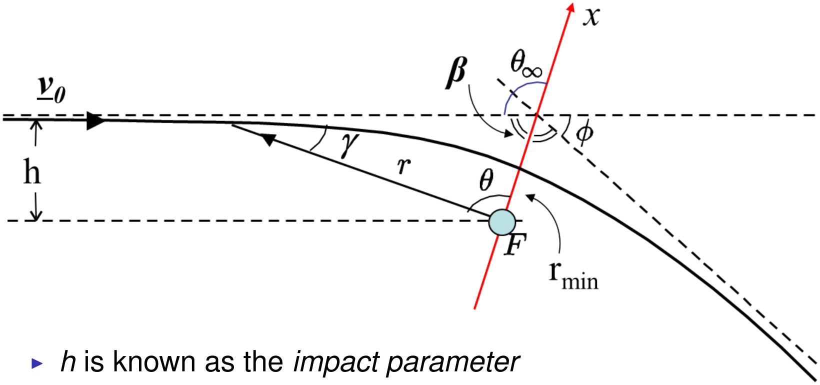


- Orbit equation : $r(\theta) = \frac{r_0}{1+e\cos\theta}$
- Ellipse : $e < 1$ $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$
- Hyperbola :
  $e > 1$ $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$

When $x\&y \to \infty$, $\frac{y}{b} = \frac{x}{a} \to y = \left(\frac{b}{a}\right)x$

20.2 Hyperbolic orbit : the distance of closest approach

For example a comet deviated by the gravitational attraction of a planet. Velocity $\underline{\mathbf{v}} = \underline{\mathbf{v}_0}$ when $\underline{\mathbf{r}} \to \infty$.



- h is known as the *impact parameter*
- Angular momentum $\underline{\mathbf{J}} = m\,\underline{\mathbf{r}} \times \underline{\mathbf{v}}$

  $\to |\underline{\mathbf{J}}| = mvr\sin\gamma = mv_0 h$ (as $r \to \infty$)

  $\to$ Total energy $E = \frac{1}{2} m v_0^2$ (again as $r \to \infty$)

Distance of closest approach continued

- $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{\alpha}{r}$, where $\alpha = GmM$
- At distance of closest approach $r = r_{min} \rightarrow \dot{r} = 0$
- $E = \frac{J^2}{2mr_{min}^2} - \frac{\alpha}{r_{min}}$

  $\rightarrow r_{min}^2 + \frac{\alpha}{E}r_{min} - \frac{J^2}{2mE} = 0$

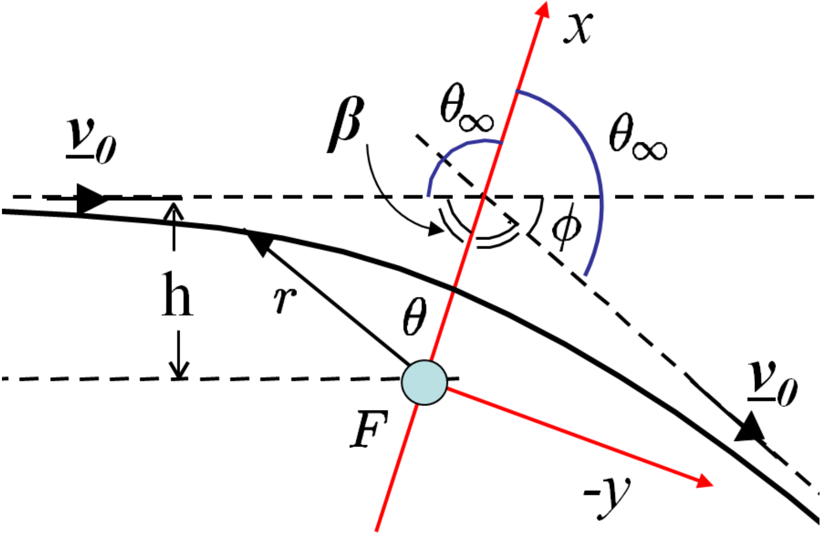
- Same form of solution as for the ellipse :
- $r_{min} = -\left(\frac{\alpha}{2E}\right)[1 - \underbrace{(1 + \frac{2EJ^2}{m\alpha^2})^{\frac{1}{2}}}_{\text{e}}]$ $\quad (J^2 = (mv_0 h)^2\ ;\ E = \frac{1}{2}mv_0^2)$

  $\rightarrow \quad r_{min} = \underbrace{-\frac{\alpha}{2E}(1 - e)}_{= \text{a(e - 1)}}$

- Velocity v' at distance of closest approach: line to trajectory is a right angle.

  $\rightarrow \ J = mv'r_{min} = mv_0 h \ \rightarrow \ v' = \frac{v_0 h}{r_{min}}$

20.3 Hyperbolic orbit: the angle of deflection, ϕ



20.3.1 Method 1 : using impulse

- Directly from the diagram : $\Delta v_x = 2v_0 \cos\theta_\infty$ (1)
- By symmetry, integrated change in $v_y = 0$: $\Delta v_y = 0$
- Change in Δp_x : $m\Delta v_x = \int_{-\infty}^{+\infty} F_x dt = \int_{-\infty}^{+\infty} F_x \underbrace{\left(\frac{mr^2\dot{\theta}}{J}\right)}_{=1} dt$

  $\rightarrow \quad m\Delta v_x = \left(\frac{m}{J}\right) \int_{-\theta_\infty}^{+\theta_\infty} F_x\, r^2 d\theta$
- But $\underline{\mathbf{F}} = -\left(\frac{\alpha}{r^2}\right)\underline{\hat{\mathbf{r}}} \quad \rightarrow \quad F_x = -\frac{\alpha}{r^2}\cos\theta$

  $\rightarrow \quad m\Delta v_x = -2\left(\frac{m\alpha}{J}\right) \int_0^{\theta_\infty} \cos\theta\, d\theta$

  $\rightarrow \quad \Delta v_x = -\left(\frac{2\alpha}{J}\right)\sin\theta_\infty$ (2)
- From (1) & (2) $\rightarrow -\left(\frac{2\alpha}{J}\right)\sin\theta_\infty = 2v_0 \cos\theta_\infty$
- $\tan\theta_\infty = -\frac{Jv_0}{\alpha} \quad \rightarrow \quad \theta_\infty + \beta = \pi \; ; \; \phi + 2\beta = \pi$
- $\theta_\infty = \frac{\phi}{2} + \frac{\pi}{2}$; $\tan\theta_\infty = \tan\left(\frac{\phi}{2} + \frac{\pi}{2}\right) = -\cot\frac{\phi}{2}$

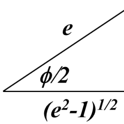
$$\boxed{\cot\frac{\phi}{2} = \frac{J\,v_0}{\alpha} = \frac{mhv_0^2}{\alpha} = \frac{hv_0^2}{GM}}$$

20.3.2 Method 2 : using hyperbola orbit parameters

- Orbit equation : $r(\theta) = \frac{r_0}{1+e\cos\theta}$
- $r \to \infty$, $\cos\theta_\infty = -\frac{1}{e}$
- $\theta_\infty + \beta = \pi$; $\phi + 2\beta = \pi \to \frac{\phi}{2} = \theta_\infty - \frac{\pi}{2} \to \sin\frac{\phi}{2} = \frac{1}{e}$

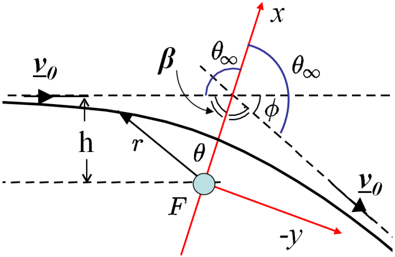
- From before :

$$r_{min} = -\left(\frac{\alpha}{2E}\right)\left[1 - \underbrace{\left(1 + \frac{2EJ^2}{m\alpha^2}\right)^{\frac{1}{2}}}_{e}\right]$$



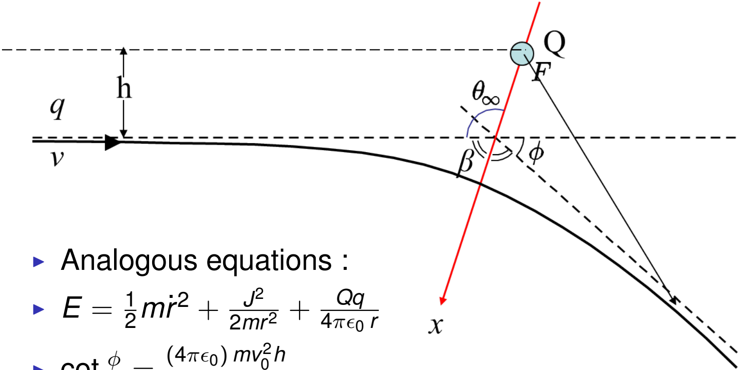
- $\cot\frac{\phi}{2} = [e^2-1]^{\frac{1}{2}} = [\frac{2EJ^2}{m\alpha^2}]^{\frac{1}{2}}$
- $E = \frac{1}{2}mv_0^2$; $J = mv_0 h$
- $\boxed{\cot\frac{\phi}{2} = \frac{mv_0^2 h}{\alpha} = \frac{v_0^2 h}{GM}}$

as before



20.4 Hyperbolic orbit : Rutherford scattering

A particle of charge $+q$ deviated by the Coulomb repulsion of a nucleus, charge $+Q$. Analogous hyperbolic motion as previously: $\alpha = GMm \;\rightarrow\; \alpha = -\frac{Qq}{4\pi\epsilon_0}$. As before, velocity $\underline{\mathbf{v}} = \underline{\mathbf{v}}_\mathbf{0}$ when $\underline{\mathbf{r}} \rightarrow \infty$. Here we take the other branch of the hyperbola.



- Analogous equations :
- $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + \frac{Qq}{4\pi\epsilon_0\, r}$
- $\cot\frac{\phi}{2} = \frac{(4\pi\epsilon_0)\, m v_0^2 h}{qQ}$
- $r_{min} = \left(\frac{qQ}{2E(4\pi\epsilon_0)}\right)[1 + (1 + \frac{2EJ^2}{m\alpha^2})^{\frac{1}{2}}]$

21.1 NII for system of particles - translation motion

Reminder from MT lectures:

- Force on particle i: $\quad m_i \frac{d^2}{dt^2}(\underline{\mathbf{r}}_\mathbf{i}) = \underline{\mathbf{F}}_\mathbf{i}{}^{ext} + \underline{\mathbf{F}}_\mathbf{i}{}^{int}$
- $$\underbrace{\sum_i^N m_i \frac{d^2}{dt^2}(\underline{\mathbf{r}}_\mathbf{i})}_{\text{all masses}} = \underbrace{\sum_i^N \underline{\mathbf{F}}_\mathbf{i}{}^{ext}}_{\text{external forces}} + \underbrace{\sum_i^N \underline{\mathbf{F}}_\mathbf{i}{}^{int}}_{\text{internal forces = zero}} = \sum_i^N \underline{\mathbf{F}}_\mathbf{i}{}^{ext}$$

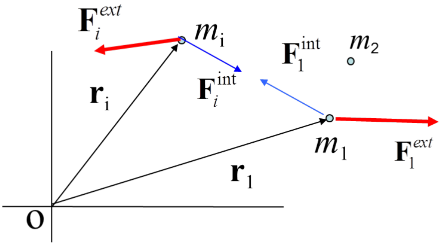
- $\underline{\mathbf{r}}_{CM} = \sum_i^N \frac{m_i\, \underline{\mathbf{r}}_\mathbf{i}}{M}$

  where $M = \sum_i^N m_i$

- $\underline{\mathbf{v}}_{CM} = \dot{\underline{\mathbf{r}}}_{CM} = \sum_i^N \frac{m_i\, \dot{\underline{\mathbf{r}}}_i}{M}$

  $\rightarrow \; \underline{\mathbf{P}}_{CM} = \sum_i^N m_i \dot{\underline{\mathbf{r}}}_\mathbf{i} = M\, \underline{\mathbf{v}}_{CM}$

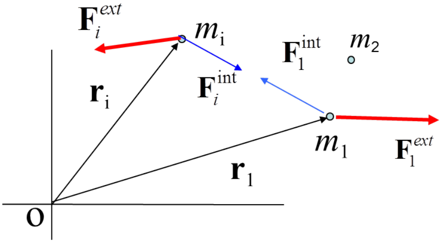
- $\underline{\mathbf{a}}_{CM} = \ddot{\underline{\mathbf{r}}}_{CM} = \sum_i^N \frac{m_i\, \ddot{\underline{\mathbf{r}}}_i}{M} = \frac{\underline{\mathbf{F}}_\mathbf{i}{}^{ext}}{M}$



21.1.1 Kinetic energy and the CM

- Lab kinetic energy : $T = \frac{1}{2}\sum_i^N m_i\, \underline{\mathbf{v}}_i^2 \quad ; \quad \underline{\mathbf{v}}_i = \underline{\mathbf{v}}'_i + \underline{\mathbf{v}}_{CM}$
  where $\underline{\mathbf{v}}'_i$ is velocity of particle i in the CM
- $T = \frac{1}{2}\sum_i m_i\, \underline{\mathbf{v}}_i'^2 + \frac{1}{2}\sum_i m_i\, \underline{\mathbf{v}}_{CM}^2 + \sum_i m_i\, \underline{\mathbf{v}}'_i \cdot \underline{\mathbf{v}}_{CM}$
- But $\sum_i m_i \underline{\mathbf{v}}'_i \cdot \underline{\mathbf{v}}_{CM} = \underbrace{\frac{\sum_i m_i \underline{\mathbf{v}}'_i}{M}}_{=0} \cdot M\, \underline{\mathbf{v}}_{CM}$
- $\boxed{T = T' + \frac{1}{2} M \underline{\mathbf{v}}_{CM}^2}$

Same expression as was derived in MT

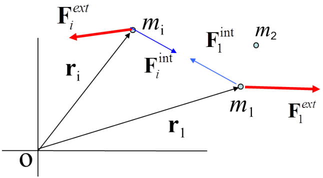


21.2 NII for system of particles - rotational motion

- Angular momentum of particle i about O : $\underline{\mathbf{J}}_i = \underline{\mathbf{r}}_i \times \underline{\mathbf{p}}_i$
- Torque of i about O: $\underline{\tau}_i = \frac{d\underline{\mathbf{J}}_i}{dt} = \underline{\mathbf{r}}_i \times \underline{\dot{\mathbf{p}}}_i + \underbrace{\underline{\dot{\mathbf{r}}}_i \times \underline{\mathbf{p}}_i}_{=0} = \underline{\mathbf{r}}_i \times \underline{\mathbf{F}}_i$

- Total ang. mom. of system

$$\underline{\mathbf{J}} = \sum_i^N \underline{\mathbf{J}}_i = \sum_i^N \underline{\mathbf{r}}_i \times \underline{\mathbf{p}}_i$$

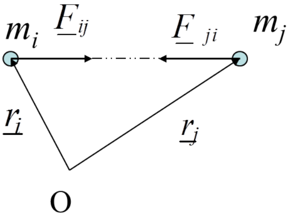


- Internal forces:

$$\sum_{pair}^{int} \underline{\tau}_{(i,j)} = \underline{\mathbf{r}}_i \times \underline{\mathbf{F}}_{ij}^{int} + \underline{\mathbf{r}}_j \times \underline{\mathbf{F}}_{ji}^{int}$$

$$= (\underline{\mathbf{r}}_i - \underline{\mathbf{r}}_j) \times \underline{\mathbf{F}}_{ij}^{int}$$

$= 0$ since $(\underline{\mathbf{r}}_i - \underline{\mathbf{r}}_j)$ parallel to $\underline{\mathbf{F}}_{ij}^{int}$



Hence total torque

$$\underline{\tau} = \sum_i^N (\underline{\mathbf{r}}_i \times \underline{\mathbf{F}}_i^{ext}) = \frac{d\underline{\mathbf{J}}}{dt}$$

If external torque $= 0$, $\underline{\mathbf{J}}$ is const.

21.2.1 Angular momentum and the CM

- Lab to CM : $\underline{\mathbf{r}}_i = \underline{\mathbf{r}}'_i + \underline{\mathbf{r}}_{CM}$; $\underline{\mathbf{v}}_i = \underline{\mathbf{v}}'_i + \underline{\mathbf{v}}_{CM}$

  where $\underline{\mathbf{r}}'_i$, $\underline{\mathbf{v}}'_i$ are position & velocity of particle i wrt the CM
- Total ang. mom. of system

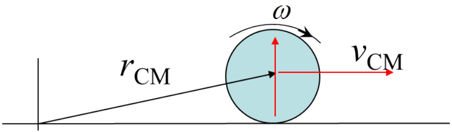
$$\underline{\mathbf{J}} = \sum_i^N \underline{\mathbf{J}}_i = \sum_i^N m_i\,(\underline{\mathbf{r}}'_i + \underline{\mathbf{r}}_{CM}) \times (\underline{\mathbf{v}}'_i + \underline{\mathbf{v}}_{CM})$$

$$= \sum_i m_i\,(\underline{\mathbf{r}}'_i \times \underline{\mathbf{v}}'_i) + \sum_i m_i\,(\underline{\mathbf{r}}'_i \times \underline{\mathbf{v}}_{CM}) + \sum_i m_i\,(\underline{\mathbf{r}}_{CM} \times \underline{\mathbf{v}}'_i) + \sum_i m_i\,(\underline{\mathbf{r}}_{CM} \times \underline{\mathbf{v}}_{CM})$$

But $\sum_i m_i\,(\underline{\mathbf{r}}'_i \times \underline{\mathbf{v}}_{CM}) = \underbrace{[\sum_i m_i\,\underline{\mathbf{r}}'_i]}_{=\text{ 0 in CM}} \times \underline{\mathbf{v}}_{CM}$; $\sum_i m_i\,(\underline{\mathbf{r}}_{CM} \times \underline{\mathbf{v}}'_i) = \underline{\mathbf{r}}_{CM} \times \underbrace{[\sum_i m_i\,\underline{\mathbf{v}}'_i]}_{=\text{ 0 in CM}}$

- Hence

$$\boxed{\underline{\mathbf{J}} = \underbrace{\underline{\mathbf{J}}'}_{\text{J wrt CM}} + \underbrace{\underline{\mathbf{r}}_{CM} \times M\,\underline{\mathbf{v}}_{CM}}_{\text{J of CM translation}}}$$



What we have learned so far

- Newton's laws relate to rotating systems in the same way that the laws relate to transitional motion.
- For any system of particles, the rate of change of internal angular momentum about an origin is equal to the total torque of the external forces about the origin.
- The total angular momentum about an origin is the sum of the total angular momentum about the CM plus the angular momentum of the translation of the CM.

21.3 Introduction to Moment of Inertia

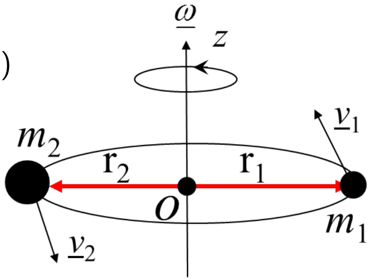
- Take the simplest example of 2 particles rotating in circular motion about a common *axis of rotation* with angular velocity $\underline{\omega} = \omega\,\hat{\underline{\mathbf{z}}}$
- Definition of $\underline{\omega}$ for circular motion : $\boxed{\dot{\underline{\mathbf{r}}} = \underline{\omega} \times \underline{\mathbf{r}}}$

- Total angular momentum of the system of particles about O

  $$\underline{\mathbf{J}} = \underline{\mathbf{r}}_1 \times (m_1 \underline{\mathbf{v}}_1) + \underline{\mathbf{r}}_2 \times (m_2 \underline{\mathbf{v}}_2)$$

- $\underline{\mathbf{v}}_1 = \underline{\omega} \times \underline{\mathbf{r}}_1$

  $\underline{\mathbf{v}}_2 = \underline{\omega} \times \underline{\mathbf{r}}_2$



- Since $\underline{\mathbf{r}}_i \perp \underline{\mathbf{v}}_i$

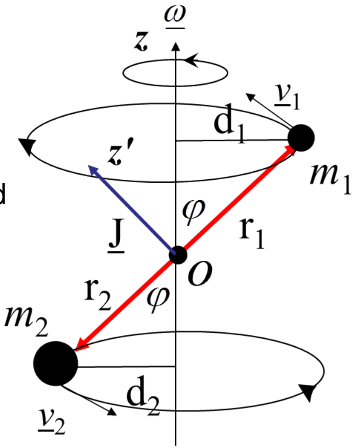
  $$\underline{\mathbf{J}} = (m_1\, r_1^2 + m_2\, r_2^2)\,\underline{\omega}$$

- $\underline{\mathbf{J}} = \mathtt{I}\underline{\omega}$ $\quad$ ($\underline{\mathbf{J}}$ is parallel to $\underline{\omega}$)
  - Moment of Inertia $\quad \mathtt{I} = m_1\, r_1^2 + m_2\, r_2^2$

    Or more generally $\; \mathtt{I} = \sum_i [m_i\, r_i^2]$

21.3.1 Extend the example : J not parallel to ω

Now consider the same system but with rotation tilted wrt rotation axis by an angle ϕ. Again $\underline{\omega} = \omega\,\underline{\hat{\mathbf{z}}}$



- Total angular momentum of the particles about O :

  $\underline{\mathbf{J}} = \underline{\mathbf{r}}_1 \times (m_1 \underline{\mathbf{v}}_1) + \underline{\mathbf{r}}_2 \times (m_2 \underline{\mathbf{v}}_2)$

- NB. $\underline{\mathbf{J}}$ now points along the $\underline{\mathbf{z}}'$ axis
- $\underline{\mathbf{J}}$ vector is $\perp$ to line of m_1 and m_2 and defines the *principal axis* of the MoI mass distribution (see later)
- Since $v_1 = d_1\,\omega \;\; ; \quad r_1 = \frac{d_1}{\sin\phi}$

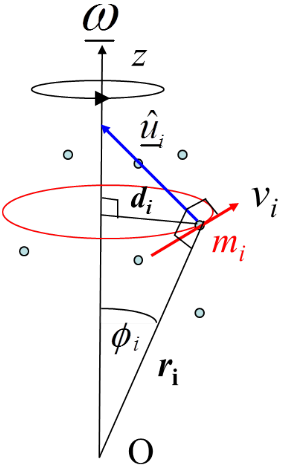
  $v_2 = d_2\,\omega \;\; ; \quad r_2 = \frac{d_2}{\sin\phi}$

- Then $\quad |\underline{\mathbf{J}}| = \frac{(m_1\,d_1^2 + m_2\,d_2^2)\,\omega}{\sin\phi}$

  $\rightarrow |\underline{\mathbf{J}}| \sin\phi\, \hat{\underline{\omega}} = \mathtt{I}_z\, \underline{\omega}$. Hence $J_z = \mathtt{I}_z \omega$ *where* $\mathtt{I}_z$ *is calculated about the* $\hat{\underline{\mathbf{z}}}$ *(i.e.* $\hat{\underline{\omega}}$*) axis.*

21.3.2 Moment of inertia : mass not distributed in a plane

Now take a system of particles rotating in circular motion about a common axis of rotation, all with angular velocity $\underline{\omega}$ (where $\underline{\mathbf{v}}_i = \underline{\omega} \times \underline{\mathbf{r}}_i$).



- Total angular momentum of the system of particles about O (which is on the axis of rotation)

  $\underline{\mathbf{J}} = \sum_i^N \underline{\mathbf{r}}_i \times (m_i \underline{\mathbf{v}}_i) = \sum_i^N m_i r_i v_i \, \underline{\hat{\mathbf{u}}}_\mathbf{i}$

  (not necessarily parallel to $\underline{\omega}$ axis)

- As before, resolve the angular momentum about the axis of rotation

  $J_z \, \underline{\hat{\mathbf{z}}} = \sum_i^N m_i r_i v_i \sin \phi_i \, \underline{\hat{\omega}}$

- $\underline{\mathbf{v}}_i = \underline{\omega} \times \underline{\mathbf{r}}_i$; $v_i = \omega r_i \sin \phi_i$

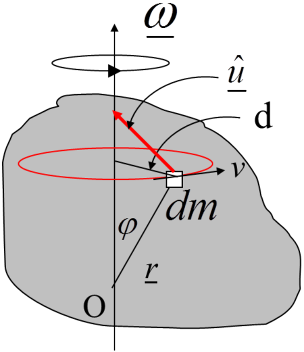
  Also $\sin \phi_i = \frac{d_i}{r_i}$; $v_i = \omega \, d_i$

$J_z \, \underline{\hat{\omega}} = \left(\sum_i^N m_i \, d_i^2 \right) \underline{\omega} \;\rightarrow\; J_z = \mathtt{I}_z \, \omega$ ($\mathtt{I}_z$: MoI about rot^n axis)

21.3.3 Generalize for rigid bodies

A rigid body may be considered as a collection of infinitesimal point particles whose relative distance does not change during motion.

- $\sum_i m_i \to \int dm,\ where\ dm = \rho dV$ and ρ is the volume density
- $\mathtt{I}_z = \left(\sum_i^N m_i\, d_i^2\right) \ \to\ \int_V d^2\, \rho\, dV$
- This integral gives the moment of inertia about axis of rotation (z axis)

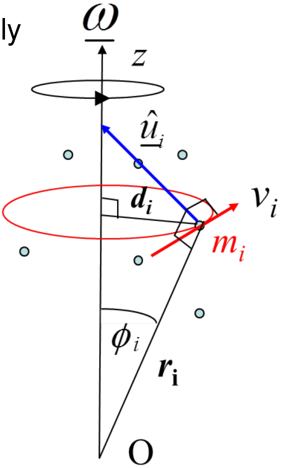


22.1 Moment of inertia tensor

- Consider bodies rotating around a common axis $\underline{\omega}$, no axis of symmetry, $\underline{\mathbf{J}}$ not necessarily parallel to $\underline{\omega}$, origin O lies on the $\underline{\omega}$ $(\underline{\mathbf{z}})$ axis.
- Definition of angular velocity : $\dot{\underline{\mathbf{r}}} = \underline{\omega} \times \underline{\mathbf{r}}$
- $\underline{\mathbf{J}} = \sum_i \underline{\mathbf{r}}_i \times \underline{\mathbf{p}}_i = \sum_i m_i \, \underline{\mathbf{r}}_i \times \dot{\underline{\mathbf{r}}}_i$
  $= \sum_i m_i \, \underline{\mathbf{r}}_i \times (\underline{\omega} \times \underline{\mathbf{r}}_i)$
- Use the vector identity
  $\underline{\mathbf{a}} \times (\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = (\underline{\mathbf{a}}.\underline{\mathbf{c}})\,\underline{\mathbf{b}} - (\underline{\mathbf{a}}.\underline{\mathbf{b}})\,\underline{\mathbf{c}}$

$$\boxed{\underline{\mathbf{J}} = \sum_i m_i \, r_i^2 \, \underline{\omega} - \sum_i m_i \, (\underline{\mathbf{r}}_i \cdot \underline{\omega}) \, \underline{\mathbf{r}}_i}$$

[Note for circular motion in a plane where $\underline{\mathbf{r}}_i$ is $\perp$ to $\underline{\omega}$, $\sum_i m_i \, (\underline{\mathbf{r}}_i \cdot \underline{\omega}) = 0$, and hence in this case $\underline{\mathbf{J}}$ is $\parallel$ to $\underline{\omega}$.]



Moment of inertia tensor continued

- From before : $\underline{\mathbf{J}} = \sum_i m_i r_i^2 \underline{\omega} - \sum_i m_i \left(\underline{\mathbf{r}}_i \cdot \underline{\omega}\right) \underline{\mathbf{r}}_i$
- Can express in terms of components

$$(J_x, J_y, J_z) = \sum_i m_i(x_i^2 + y_i^2 + z_i^2)\,(\omega_x, \omega_y, \omega_z) - \sum_i m_i(x_i\,\omega_x + y_i\,\omega_y + z_i\,\omega_z)\,(x_i, y_i, z_i)$$

- Construct a matrix equation :

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \sum_i(y_i^2 + z_i^2)m_i & -\sum_i(x_i y_i)m_i & -\sum_i(x_i z_i)m_i \\ -\sum_i(x_i y_i)m_i & \sum_i(x_i^2 + z_i^2)m_i & -\sum_i(y_i z_i)m_i \\ -\sum_i(x_i z_i)m_i & -\sum_i(y_i z_i)m_i & \sum_i(x_i^2 + y_i^2)m_i \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

If mass is continuous (rigid body) :

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \int(y^2 + z^2)\,dm & -\int xy\,dm & -\int xz\,dm \\ -\int xy\,dm & \int(x^2 + z^2)\,dm & -\int yz\,dm \\ -\int xz\,dm & -\int yz\,dm & \int(x^2 + y^2)\,dm \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

- Hence we can write : $\underline{\mathbf{J}} = \widetilde{\mathtt{I}}\underline{\omega}$
- $\widetilde{\mathtt{I}}$ is the *Moment of Inertia tensor* of the system

22.1.1 Rotation about a principal axis

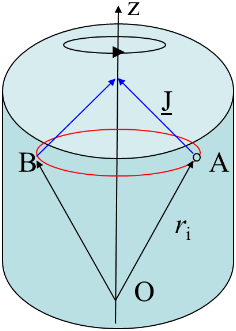
- In general $\underline{\mathbf{J}} = \widetilde{\mathrm{I}}\,\underline{\omega}$, where $\widetilde{\mathrm{I}}$ is the *Moment of Inertia Tensor*

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \mathrm{I}_{xx} & \mathrm{I}_{xy} & \mathrm{I}_{xz} \\ \mathrm{I}_{yx} & \mathrm{I}_{yy} & \mathrm{I}_{yz} \\ \mathrm{I}_{zx} & \mathrm{I}_{zy} & \mathrm{I}_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

- Whenever possible, one aligns the axes of the coordinate system in such a way that the mass of the body evenly distributes around the axes: we choose *axes of symmetry*.

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \mathrm{I}_{xx} & 0 & 0 \\ 0 & \mathrm{I}_{yy} & 0 \\ 0 & 0 & \mathrm{I}_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

The diagonal terms are called the *principal axes* of the moment of inertia.



- Whenever we rotate about an axis of symmetry, for every point A there is a point B which cancels it, and

$$\boxed{\underline{\mathbf{J}} \rightarrow J_z\,\underline{\hat{\mathbf{z}}} = \mathrm{I}_{zz}\,\omega\,\underline{\hat{\mathbf{z}}}}$$

and where $\underline{\mathbf{J}}$ is parallel to $\underline{\omega}$ along the $\underline{\mathbf{z}}$ axis

22.2 Moment of inertia & energy of rotation

Particles rotating in circular motion about a common axis of rotation with angular velocity $\underline{\omega}$ (where $\underline{\mathbf{v}}_i = \underline{\omega} \times \underline{\mathbf{r}}_i$).

- Kinetic energy of mass m_i :

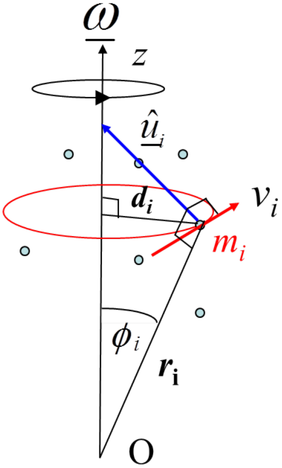
  $T_i = \frac{1}{2} m_i v_i^2$
- Total KE = $\frac{1}{2} \sum_i \left(m_i v_i^2\right)$
- $\underline{\mathbf{v}}_i = \underline{\omega} \times \underline{\mathbf{r}}_i$

  $v_i = \omega \, r_i \, \sin \phi_i$

  $\sin \phi_i = \frac{d_i}{r_i}$
- $T_{rot} = \frac{1}{2} \left(\sum_i^N m_i \, d_i^2 \right) \omega^2$

  $$\boxed{T_{rot} = \frac{1}{2} \mathtt{I}_z \, \omega^2}$$

  where $\mathtt{I}_z$ is calculated about the axis of rotation



22.3 Calculation of moments of inertia

22.3.1 MoI of a thin rectangular plate

(a) About the x axis

- $\mathtt{I}_x = \int y^2\, dm$

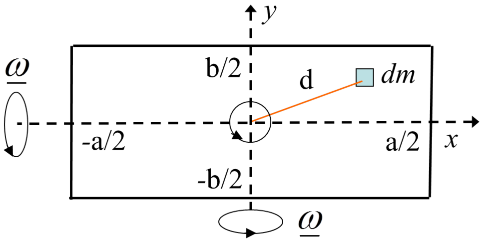
$dm = \rho\, dx\, dy\ ;\ \rho = \frac{M}{ab}$

- $\mathtt{I}_x = \int y^2\, \rho\, dx\, dy$

$$= \left[\rho\, \frac{y^3}{3}\right]_{-\frac{b}{2}}^{+\frac{b}{2}} [x]_{-\frac{a}{2}}^{+\frac{a}{2}}$$

$$= \rho\, a \left[\frac{b^3}{24} + \frac{b^3}{24}\right] = \rho\, a\, b \left[\frac{b^2}{12}\right]$$

- Hence $\boxed{\mathtt{I}_x = \frac{M\, b^2}{12}}$



(b) About the y axis

- $\boxed{\mathtt{I}_y = \frac{M\, a^2}{12}}$

(c) About the z axis

- $\mathtt{I}_z = \int \rho\, d^2\, dx\, dy$

$$= \int \rho\, (x^2 + y^2)\, dx\, dy$$

$$= \left[\rho\, \frac{y^3}{3}\right]_{-\frac{b}{2}}^{+\frac{b}{2}} + \left[\frac{x^3}{3}\right]_{-\frac{a}{2}}^{+\frac{a}{2}}$$

- Hence $\boxed{\mathtt{I}_z = M\, (\frac{a^2 + b^2}{12})}$

22.3.2 MoI of a thin disk perpendicular to plane of disk

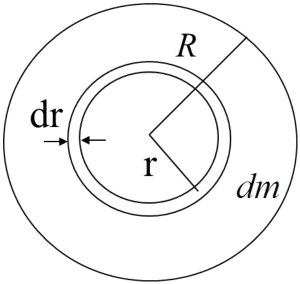
- $\mathtt{I}_z = \int r^2\, dm$

$dm = \rho(2\pi r\, dr)$; $\rho = \frac{M}{\pi R^2}$

- $\mathtt{I}_z = 2\pi\rho \int r^3 dr$

$$= \left[2\pi\rho\, \frac{r^4}{4}\right]_0^R = (\pi\, \frac{R^4}{2})\, \frac{M}{\pi R^2}$$

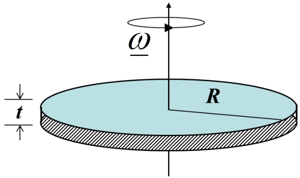
- Hence $\boxed{\mathtt{I}_z = \frac{1}{2} M\, R^2}$



- For a cylinder thickness t :
- Use cylindrical coordinates

$dm = \rho(2\pi r\, dr\, dt)$; $\rho = \frac{M}{\pi R^2\, t}$

- $\mathtt{I}_z = 2\pi\rho \int \int r^3 dr\, dt$
- $\boxed{\mathtt{I}_z = \frac{1}{2} M\, R^2}$, the same.



22.3.3 MoI of a solid sphere

- $d\mathtt{I} = \frac{1}{2}\, x^2\, dm$ (for a disk)

$$dm = \rho(\pi x^2\, dz)\ ;\ \rho = \frac{M}{\frac{4}{3}\pi R^3}$$

$$\rightarrow \quad d\mathtt{I} = \frac{\pi\rho}{2}\, x^4\, dz$$

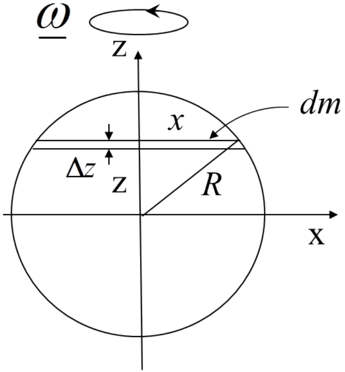
- $\mathtt{I} = \int d\mathtt{I} = \frac{1}{2}\pi\rho \int_{-R}^{+R} x^4 dz$
- But $x^2 = R^2 - z^2$

$$\mathtt{I} = 2 \times \frac{1}{2}\pi\rho \int_0^{+R} (R^4 - 2R^2 z^2 + z^4) dz$$

$$= \pi\rho \left[R^4 z - \frac{2R^2 z^3}{3} + \frac{1}{5} z^5 \right]_0^R$$

$$= \pi\rho\, \frac{8}{15} R^5 = M(\pi\, \frac{8}{15} R^5)/(\frac{4}{3}\pi R^3)$$

- Hence $\boxed{\mathtt{I} = \frac{2}{5} M\, R^2}$

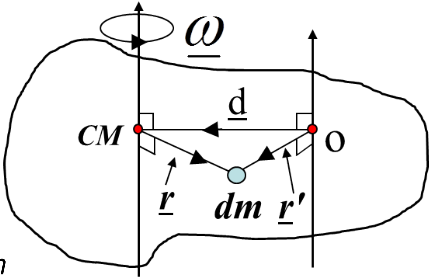


23.1 Parallel axis theorem

$\mathtt{I}_{CM}$ is the moment of inertia of body mass M about an axis passing through its centre of mass. $\mathtt{I}$ is the moment of inertia of the body about a parallel axis a distance d from the first.

- About the axis through the CM

  $\mathtt{I}_{CM} = \int r^2 dm$
- $\underline{\mathbf{r}}' = \underline{\mathbf{d}} + \underline{\mathbf{r}}$
- $r'^2 = d^2 + 2\,\underline{\mathbf{d}} \cdot \underline{\mathbf{r}} + r^2$
- About the parallel axis :
  $\mathtt{I} = \int r'^2 dm$



$$= \int d^2 dm + 2\,\underline{\mathbf{d}} \cdot \underbrace{\int \underline{\mathbf{r}}\, dm}_{=0} + \int r^2 dm$$

(definition of CM)

- Hence $\boxed{\mathtt{I} = \mathtt{I}_{CM} + Md^2}$

23.1.1 Example : compound pendulum

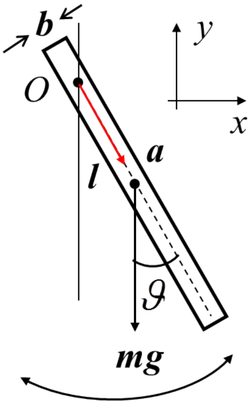
Rectangular rod length a width b mass m swinging about axis O, distance ℓ from the CM, in plane of paper

- $\mathtt{I}_{CM} = m\left(\frac{a^2+b^2}{12}\right)$
- Parallel axis theorem :

  $\mathtt{I} = \mathtt{I}_{CM} + m\ell^2$
- Torque about O $= \underline{\mathbf{r}} \times \underline{\mathbf{F}}$:

  $\underline{\tau} = -m\,g\,\ell \sin\theta\,\underline{\hat{\mathbf{k}}}$
- $\underline{\mathbf{J}} = \mathtt{I}\,\underline{\omega} = \mathtt{I}\,\dot{\theta}\,\underline{\hat{\mathbf{k}}}$
- Differentiate : $\underline{\tau} = \frac{d\mathbf{J}}{dt} = \mathtt{I}\,\ddot{\theta}\,\underline{\hat{\mathbf{k}}}$
- Equate $\mathtt{I}\,\ddot{\theta} = -m\,g\,\ell \sin\theta$
- Small angle approximation

  $\ddot{\theta} + \frac{m\,g\,\ell}{\mathtt{I}}\theta = 0$

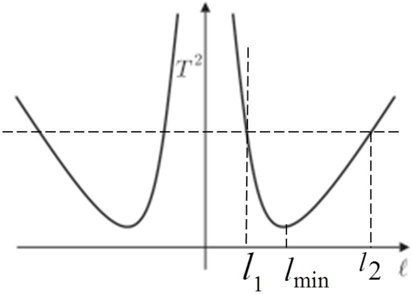


Compound pendulum continued

- $\ddot{\theta} + \frac{m\,g\,\ell}{\mathtt{I}}\theta = 0$

  where $\mathtt{I} = m\,(\frac{a^2+b^2}{12}) + m\,\ell^2$
- SHM with period $T = 2\pi\sqrt{\frac{\mathtt{I}}{mg\ell}}$

$$\rightarrow \quad T = 2\pi\sqrt{\frac{a^2+b^2+12\ell^2}{12g\ell}}$$



- Measurement of g :
- Plot T^2 vs ℓ
- Find values of ℓ_1 and ℓ_2 that give the same value of $T \rightarrow T_1 = T_2$

Can show : $\frac{T_1^2}{(\ell_1+\ell_2)} = \frac{4\pi^2}{g}$

23.2 Perpendicular axis theorem

Consider a rigid object that *lies entirely within a plane*. The perpendicular axis theorem links $\mathtt{I}_z$ (MoI about an axis perpendicular to the plane) with $\mathtt{I}_x$, $\mathtt{I}_y$ (MoI about two perpendicular axes lying within the plane).

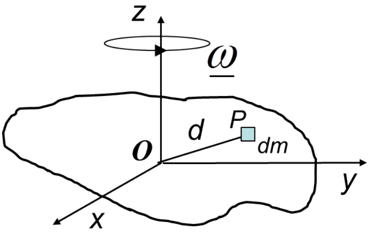
- Consider perpendicular axes $x\,,y\,,z$ (which meet at origin O) ; the body lies in the xy plane
- $\mathtt{I}_z = \int d^2\,dm$

  $= \int \left(x^2 + y^2\right)\ dm$

  $= \int x^2\,dm + \int y^2\,dm$

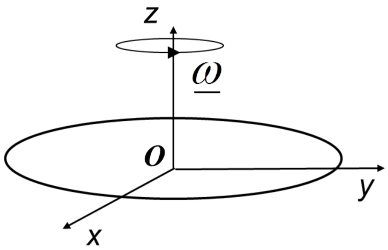
  $\rightarrow \boxed{\mathtt{I}_z = \mathtt{I}_x + \mathtt{I}_y}$

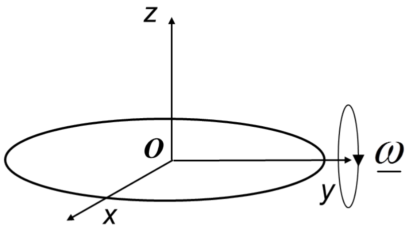
  This is the *perpendicular axis theorem*.



23.2.1 Perpendicular axis theorem : example

- Consider a thin circular disk lying in a plane
- Perpendicular axis theorem : $\mathtt{I}_z = \mathtt{I}_x + \mathtt{I}_y$
- $\mathtt{I}_z = \frac{1}{2} M R^2$
  (R = radius of disk)
- Hence $\mathtt{I}_x = \mathtt{I}_y = \frac{1}{4} M R^2$
  (due to symmetry)





23.3 Example 1 : solid ball rolling down slope

[Energy of ball] = [Rotational KE in CM] + [KE of CM] + [PE]

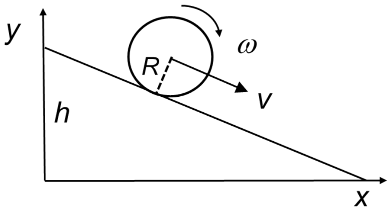
$$E = \frac{1}{2}\mathrm{I}\omega^2 + \frac{1}{2}Mv^2 + Mgy$$

- Ball falls a distance h from rest $\rightarrow$ at $y = 0$:

$$M\,g\,h = \frac{1}{2}\mathrm{I}\omega^2 + \frac{1}{2}Mv^2$$
$$= \frac{1}{2}\mathrm{I}\left(\frac{v}{R}\right)^2 + \frac{1}{2}Mv^2$$

- Solid sphere: $\mathrm{I} = \frac{2}{5}M\,R^2$
- $Mgh = \frac{1}{2}Mv^2\left(\frac{2}{5}+1\right)$

$$\rightarrow \quad v = \sqrt{\frac{10}{7}gh}$$

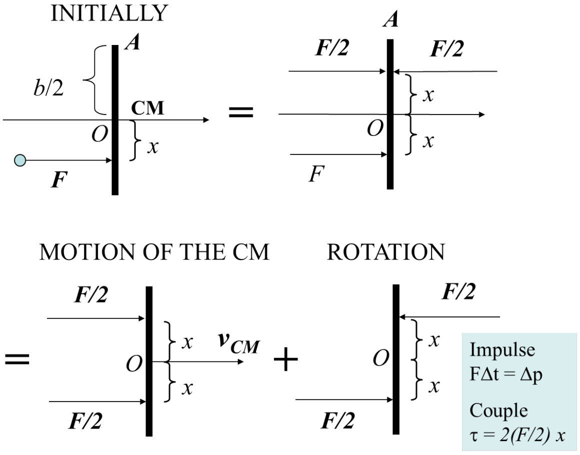


Compare with a solid cylinder $\mathrm{I} = \frac{1}{2}M\,R^2 \rightarrow v = \sqrt{\frac{4}{3}gh}$

The ball gets to the bottom faster !

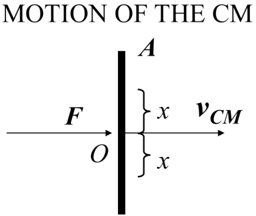
23.4 Example 2 : where to hit a ball with a cricket bat

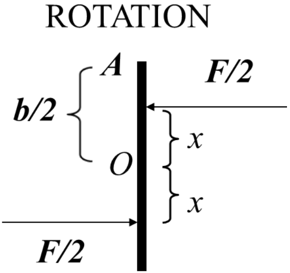
We want the bat handle (Point A, $\frac{b}{2}$ from the CM) to remain stationary after the ball has hit. When the ball hits, there is rotation of the bat about the CM, plus motion of the CM of the bat. The velocity to the right (v_{CM}) must equal the velocity from the left $\frac{\omega b}{2}$ due to rotation.



Where to hit a ball with a cricket bat continued

- Ball hits at point x from centre
- Force to the CM : $F = ma \ \rightarrow a = \frac{F}{m}$
- Moment of inertia wrt CM :
  $\mathrm{I}_{CM} = \frac{1}{12} mb^2$
- Torque (couple) about O $= \mathrm{I}_{CM}\, \ddot{\theta}$
- Hence $\mathrm{I}_{CM}\, \ddot{\theta} = \frac{mb^2}{12} \ddot{\theta} = x \times \frac{F}{2} \times 2$
  $\rightarrow \ddot{\theta} = \frac{12Fx}{mb^2}$
- Require acceleration at A to be zero.
- Acceleration at A due to rotation = $\frac{b}{2}\ddot{\theta}$
- Equate accelerations :
  $$\frac{F}{m} = \frac{6Fx}{mb} \rightarrow \ \ x = \frac{b}{6}$$





Need to hit the bat $\frac{2}{3}$ from the top

23.5 Example 3 : an aircraft landing

The landing wheel of an aircraft may be approximated as a uniform circular disk of diameter 1 m and mass 200 kg. The total mass of the aircraft including that of the 10 wheels is 100, 000 kg. When landing the touch-down speed is 50 ms^{-1}. Assume that the wheels support 50% of the total weight of the aircraft.

Determine the time duration of wheel-slip if the coefficient of friction between the wheels and the ground is 0.5*, assuming that the speed of the plane is not changed significantly.*

An aircraft landing continued

- Torque $|\underline{\tau}| = |\underline{\mathbf{r}} \times \underline{\mathbf{F}}| = a\mu M'g$ about O, where $M' = \frac{M}{20}$ (ie. 10 wheels, supporting 50% of mass)
- Angular momentum $J = \mathtt{I}\omega$ where $\mathtt{I} = \frac{1}{2}ma^2$ (MoI of solid disk where m is mass of a wheel)
- $\tau = \frac{dJ}{dt} = \mathtt{I}\frac{d\omega}{dt}$ where $u = a\omega$

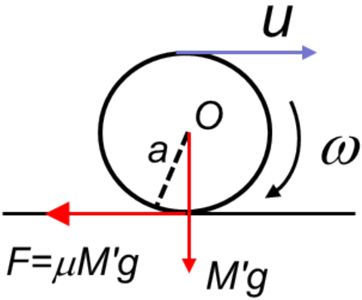
  u is the speed of the wheel rim

  $\rightarrow \tau = \frac{\mathtt{I}}{a}\frac{du}{dt} = a\mu M'g$
- Integrate : $\int_0^{t_f} \frac{a^2\mu M'g}{\mathtt{I}} dt = \int_0^{v_0} du$

  v_0 is the speed of the aeroplane

  $$\rightarrow \boxed{t_f = \frac{v_0\,\mathtt{I}}{a^2\mu M'g} = \frac{v_0\,m}{2\mu M'g}}$$

  - Putting in numbers: $t_f = \frac{50\times200}{2\times0.5\times\frac{1\times10^5}{20}\times9.8}$ $\boxed{\sim 0.2\,\mathrm{s}}$



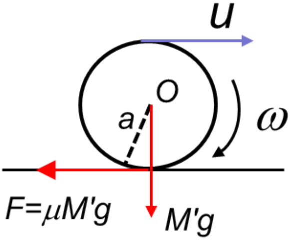
An aircraft landing continued

Confirm the assumption that the speed of the plane is not changed by calculating the speed at the end of wheel-slip in the absence of other braking processes.

- Energy expended in getting the wheels up to speed: $E_{wheels} = \frac{1}{2}\mathrm{I}\omega^2 \times 10$ (ie. 10 wheels) $= \frac{5}{2}mv_0^2$
  [Remember $v_0 = a\omega$, $\mathrm{I} = \frac{1}{2}ma^2$]
- Total energy of the aeroplane
  $E = \frac{1}{2}Mv^2$
  $\rightarrow$ Energy loss: $\delta E = Mv\,\delta v = \frac{5}{2}mv_0^2$

  $\rightarrow \delta v = \dfrac{\frac{5}{2}mv_0^2}{Mv_0} \;\rightarrow\; \dfrac{\delta v}{v_0} = \dfrac{\frac{5}{2}m}{M}$
- Putting in numbers : $\dfrac{\delta v}{v_0} = \dfrac{\frac{5}{2}\times 200}{1\times 10^5}$

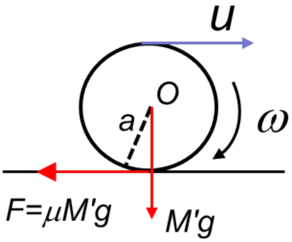
$\rightarrow$ a 0.5% effect



An aircraft landing continued

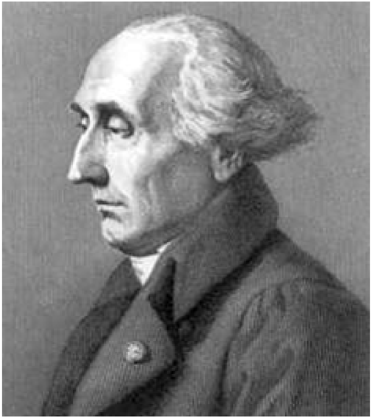
Calculate the work done during wheel slip.

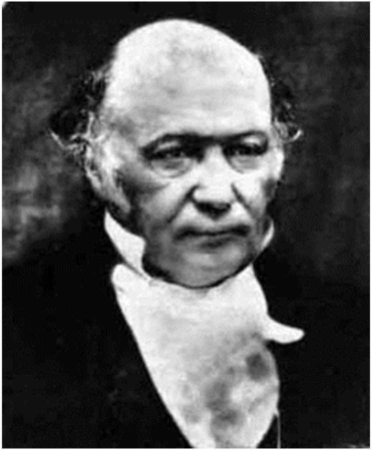
- Work done : $W = \int \tau \, d\theta = \tau \, \theta_f$
  (τ is constant, θ_f is the total turning angle)
- From before : $u = a\frac{d\theta}{dt} = \frac{2\mu M' g}{m} t$
- Integrate : $\int_0^{\theta_f} a \, d\theta = \int_0^{t_f} \frac{2\mu M' g}{m} t \, dt$
  $\rightarrow a\,\theta_f = \frac{\mu M' g}{m} t_f^2$ where $t_f = \frac{v_0 \, m}{2\mu M' g}$



- Putting it all together:
  - $$W = \tau\,\theta_f = \underbrace{\tfrac{1}{a}(a\mu M' g)}_{\tau}\underbrace{\left(\frac{M' g \mu}{m}\right)\left(\frac{v_0 \, m}{2\mu M' g}\right)^2}_{\theta_f} = \tfrac{1}{2}\mathrm{I}\left(\tfrac{v_0}{a}\right)^2$$
  - Hence $\boxed{W = E_{wheel} = \frac{1}{2}\mathrm{I}\,\omega^2}$ as expected !

Lagrange and Hamilton





- Joseph-Louis Lagrange (1736-1810)
- Sir William Rowan Hamilton (1805-1865)

24.1 Lagrangian mechanics : Introduction

- Lagrangian Mechanics: a very effective way to find the equations of motion for complicated dynamical systems using a scalar treatment

  $\rightarrow$ Newton's laws are vector relations. The Lagrangian is a single scalar function of the system variables
- Avoid the concept of force

  $\rightarrow$ For complicated situations, it may be hard to identify all the forces, especially if there are constraints
- The Lagrangian treatment provides a framework for relating conservation laws to symmetry
- The ideas may be extended to most areas of fundamental physics (special and general relativity, electromagnetism, quantum mechanics, quantum field theory ....)

24.2 Introductory example : the energy method for the E of M

- For conservative forces in 1D motion :

  Energy of system : $E = \frac{1}{2}m\dot{x}^2 + U(x)$ [Note $\frac{dE}{dt} = 0$]
- Differentiate wrt time: $m\dot{x}\ddot{x} + \frac{\partial U}{\partial x}\,\dot{x} = 0$

  $\rightarrow\ m\,\ddot{x} = -\frac{\partial U}{\partial x} = F$

  $\rightarrow$ This is the E of M for a conservative force
- Take a simple 1d spring undergoing SHM :

  $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 =$ constant

  $\frac{dE}{dt} = 0 \ \rightarrow\ m\dot{x}\ddot{x} + kx\dot{x} = 0$

  $\rightarrow\ m\ddot{x} + kx = 0$
- Hence we derived the E of M without using NII directly.

24.3 Becoming familiar with the jargon

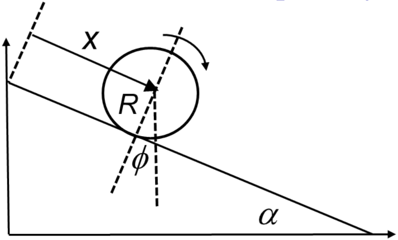
24.3.1 Generalised coordinates

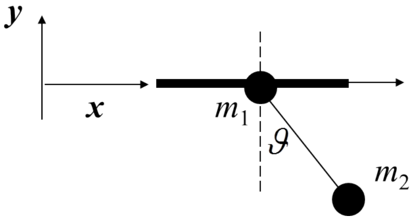
A set of parameters $q_k(t)$ that specifies the system configuration. q_k may be a geometrical parameter, x, y, z, a set of angles $\cdots$ etc

24.3.2 Degrees of Freedom

The number of degrees of freedom is the number of independent coordinates that is sufficient to describe the configuration of the system uniquely.

Examples of degrees of freedom





- Ball rolling down an incline

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\mathrm{I}\dot{\phi}^2 - mgx\sin\alpha$$

- But $\dot{x} = R\dot{\phi} \rightarrow x = R\phi$
- The problem is reduced to a 1-coordinate variable

  $q_1 \equiv x$ and $\dot{q}_1 \equiv \dot{x}$
- System has only 1 degree of freedom : x

- Pendulum whose pivot can move freely in x direction
- Pivot coordinates : $(x, 0)$
- Pendulum coordinates : $(x + \ell\sin\theta, -\ell\cos\theta)$

$$E = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\frac{d}{dt}(x + \ell\sin\theta)\right)^2 + \\ +\frac{1}{2}m_2\left(\frac{d}{dt}(-\ell\cos\theta)\right)^2 - mg\ell\cos\theta$$

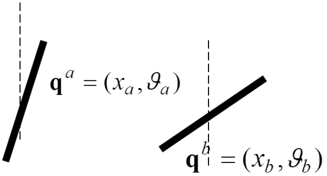
- This system has 2 degrees of freedom : x and θ

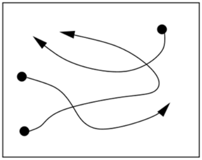
24.3.3 Constraints

- A system has *constraints* if its components are not permitted to move freely in 3-D. For example :
  - $\rightarrow$ A particle on a table is restricted to move in 2-D
  - $\rightarrow$ A mass on a simple pendulum is restricted to oscillate at an angle θ at a fixed length ℓ from a pivot
- The constraints are *Holonomic* if :
  - $\rightarrow$ The constraints are time independent
  - $\rightarrow$ The system can be described by relations between general coordinate variables and time
  - $\rightarrow$ The number of general coordinates is reduced to the number of degrees of freedom

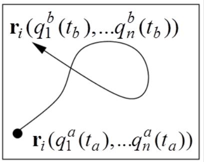
24.3.4 Configuration Space

- The configuration space of a mechanical system is an n-dimensional space whose points determine the spatial position of the system in time. This space is parametrized by generalized coordinates, $\mathbf{q} = (q_1 \cdots, q_n)$
- Example 1. A point in space determines where the system is; the coordinates are simply standard Euclidean coordinates: $(x, y, z) = (q_1, q_2, q_3)$
- Example 2. A rod location x, angle θ - as it moves in 2D space is passes through points (x, θ) in the configuration space





Real space



Configuration space

25.1 The Lagrangian : simplest illustration

The Lagrangian : $L = T - U$

- In 1D : Kinetic energy $T = \frac{1}{2}m\dot{x}^2$ No explicit dependence on x
  Potential energy $U = U(x)$ No explicit dependence on $\dot{x}$
- Define the Lagrangian in 1D : $L = \frac{1}{2}m\dot{x}^2 - U(x)$
- $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$ and $\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x}$ gives force F
- Differentiate wrt time : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} = F$
- Hence we get the Euler - Lagrange equation for x :
  $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$
- Now generalize : the Lagrangian becomes a function of $2n$ variables (n is the dimension of the configuration space). Variables are the positions and velocities
  $L(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n)$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}}$$

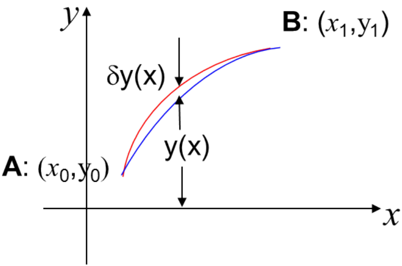
Next we expand on this concept.

25.2 The calculus of variations

- Take 2 points $A(x_0, y_0)$ and $B(x_1, y_1)$
- Curve joining them is represented by equation $y = y(x)$ such that $y(x)$ satisfies the boundary conditions :

  $\rightarrow\ y(x_0) = y_0\ ,\ y(x_1) = y_1$
- We want to find the function $y = y(x)$ subject to the above conditions which makes the closest path between the points a minimum.

  (note that this differs from what we are used to. We are not minimizing a set of variables here but a *function*).



- This is the *calculus of variations*. A branch of mathematics that deals with *functionals* as opposed to functions.

The calculus of variations, continued (1)

- We assume the unknown function f is a continuously differentiable scalar function, and the functional to be minimized depends on $y(x)$ and at most upon its first derivative $y'(x)$.
- We then wish to find the stationary values of the path between points: an integral of the form $I = \int_{x_0}^{x_1} f(y, y', x)dx$

  $\rightarrow$ $f(y, y', x)$ is a function of x, y and y' (the first derivative of y)

- Consider a small change $\delta y(x)$ in the function $y(x)$ subject to the conditions that the endpoints are unchanged :

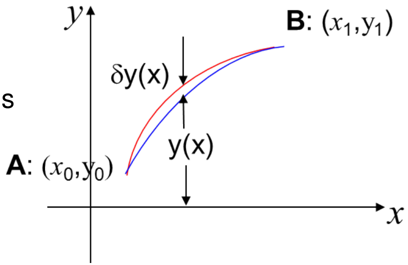
  $\rightarrow$ $\delta y(x_0) = 0$ and $\delta y(x_1) = 0$

- To first order, the variation in $f(y, y', x)$ is $\delta f = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' + \mathcal{O}(\delta y^2, \delta y'^2)$

  where $\delta y' = \frac{d}{dx}\delta y$

- Thus the variation in the integral I is

$$\delta I = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\frac{d}{dx}\delta y\right] dx$$



The calculus of variations, continued (2)

- $\delta I = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \frac{d}{dx} \delta y \right] \, dx$
- Integrate the second term by parts

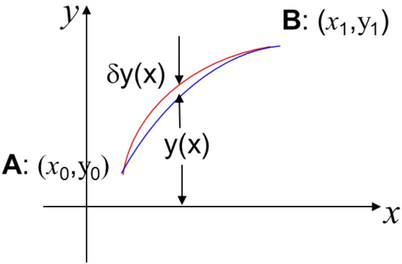
$$\text{2nd term} = \left[\frac{\partial f}{\partial y'} \delta y \right]_{x_0}^{x_1} - \int_{x_0}^{x_1} \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \delta y \, dx$$

The $\left[\frac{\partial f}{\partial y'} \delta y \right]_{x_0}^{x_1}$ term $= 0$ due to the conditions on the end points

- Hence

$$\delta I = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right] \delta y \, dx$$

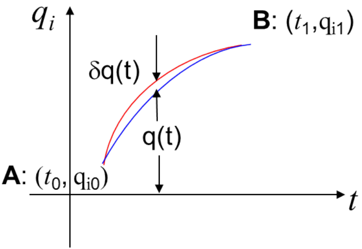
- For I to be stationary, $\delta I = 0$ for any small arbitrary variation $\delta y(x)$
  - This is only possible if the integrand vanishes identically
  - Hence we get out the Euler-Lagrange Equation



$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0}$$

The calculus of variations, continued (3)

- So far we have used x as the independent variable with a functional f which is a function of $(y(x), y', x)$
- Throughout we could have used *other* variables, in particular time t and generalized coordinates $q_1, \cdots, q_n$ and derivatives $\dot{q}_1, \cdots, \dot{q}_n$. The principles would have been the same.
- The integral $I = \int_{t_0}^{t_1} f\left[q_1(t), \cdots, q_n(t), \dot{q}_1(t), \cdots, \dot{q}_n(t)\right] dt$ must be stationary wrt variations in any one & all of the variables $q_1(t), \cdots, q_n(t)$ subject to the conditions $\delta q_i(t_0) = \delta q_i(t_1) = 0$



  - We get the n Euler-Lagrange equations for $i = 1, \cdots, n$

$$\boxed{\frac{\partial f}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right) = 0}$$

The E-L equations give the conditions for the closest path between points

25.3 A sanity check

The shortest distance between 2 points.

- Consider 2 neighboring points on the curve $y(x)$ subject to boundary conditions $y(x_0) = y_0$, $y(x_1) = y_1$
- Distance between the points $d\ell = \sqrt{dx^2 + dy^2}$
- $d\ell = \sqrt{1 + y'^2}\, dx \qquad f \equiv \sqrt{1 + y'^2}$
- The Euler-Lagrange Equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$
- $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$, $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$
- Hence $\frac{y'}{\sqrt{1+y'^2}}$ = constant, hence y' is constant
  $\rightarrow \quad y = mx + c$
- We have proved that the shortest distance between 2 points is a straight line !

25.4 Fermat's Principle & Snell's Law

Fermat : *The actual path that a light ray propagating between one point to another will take is the one that makes the time travelled between the two points stationary.*

Question: at which point $(x, 0)$ will the ray hit the interface between the two media to propagate from A to B?

- Time taken from A to B :

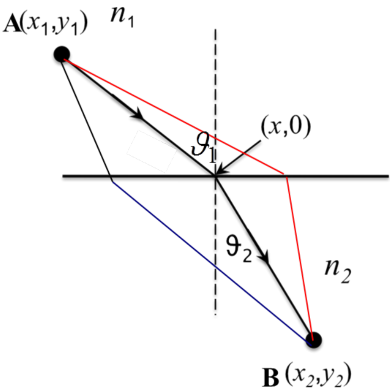
$$t(x) = \frac{1}{v_1}\left[(x - x_1)^2 + y_1^2\right]^{\frac{1}{2}} + \frac{1}{v_2}\left[(x_2 - x)^2 + y_2^2\right]^{\frac{1}{2}}$$

- The Euler-Lagrange Equation

$$\frac{\partial t}{\partial x} - \frac{d}{dy}\left(\frac{\partial t}{\partial x'}\right) = 0 \quad \text{(where the second term = 0)}$$

- $\frac{\partial t}{\partial x} = 0 = \frac{1}{v_1}\frac{x - x_1}{\left[(x-x_1)^2+y_1^2\right]^{\frac{1}{2}}} - \frac{1}{v_2}\frac{x_2 - x}{\left[(x_2-x)^2+y_2^2\right]^{\frac{1}{2}}}$
- $\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0$

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2} \quad \text{Snell's Law}$$



25.5 Hamilton's principle (Principle of Stationary Action)

- Consider for example a particle of mass m at point (x_A, y_A) moving under the influence of a force in the $x - y$ plane. We want to find the path that the particle will follow to reach a point (x_B, y_B).
- Hamilton's principle: *the path that the particle will take from A to B is the one that makes the following functional stationary* :

  $I = \int_A^B L\,(q_1(t), \cdots, q_n(t), \dot{q}_1(t), \cdots, \dot{q}_n(t))\, dt$

  where L is the Lagrangian, I is called the *action integral*
- Hence the action integral I is stationary under arbitrary variations $q_1(t), q_2(t) \cdots$ which vanish at the limits of integration ie. A and B.

26.1 Conjugate momentum and cyclic coordinates

- The E-L equation is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}$ with $L = T - U$
- Define *conjugate (generalized) momentum* : $p_k = \frac{\partial L}{\partial \dot{q}_k}$

  Note this is not necessarily linear momentum !

  $\rightarrow$ eg. simple pendulum $L = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos\theta$

  $\rightarrow$ $\frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta}$: which is angular momentum
- Following on, E-L equation reads $\dot{p}_k = \frac{\partial L}{\partial q_k}$
- If the Lagrangian L does not explicitly depend on q_k, the coordinate q_k is called *cyclic* or *ignorable*
- With no q_k dependence :

  $\frac{\partial L}{\partial q_k} = 0$ and $p_k = \frac{\partial L}{\partial \dot{q}_k}$ = constant

The momentum conjugate to a cyclic coordinate is a constant of motion

26.2 Example : rotating bead

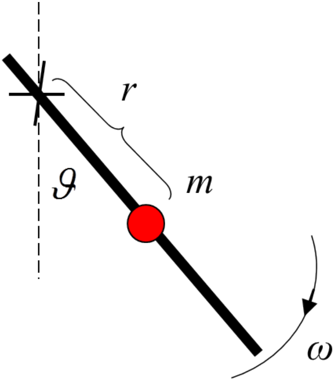
A bead slides on a wire rotating at constant angular speed ω in a horizontal plane

- Polar coordinates $\underline{\mathbf{v}} = \dot{r}\,\underline{\hat{\mathbf{r}}} + r\dot{\theta}\underline{\hat{\theta}}$
- $L = T - U$ with $U = 0$
- $L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2$
- Single variable $q_k \to r$
- E-L $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$

  $\frac{\partial L}{\partial \dot{r}} = m\dot{r} \;\to\; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$

  $\frac{\partial L}{\partial r} = mr\omega^2$
- E-L $\to m\ddot{r} - mr\omega^2 = 0$

  Central force $F_{central} = m\omega^2 r$
- $r = Ae^{\omega t} + Be^{-\omega t}$



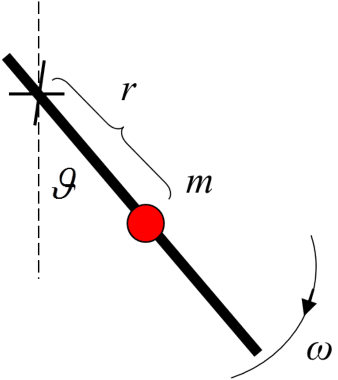
Example : rotating bead continued

What happens if the angular speed is now a free coordinate ?

- $L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$
- Two variables $q_k \rightarrow r,\ \theta$
- r variable: as before

  $\rightarrow m\ddot{r} - mr\dot{\theta}^2 = 0$
- θ variable: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$
- $\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$
- $\frac{\partial L}{\partial \theta} = 0$
- E-L : $mr^2\ddot{\theta} = \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$

  $\rightarrow$ Conservation of angular momentum

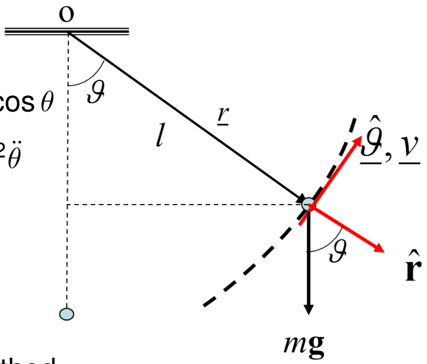


26.3 Example : simple pendulum

Evaluate simple pendulum using Euler-Lagrange equation

- Single variable $q_k \to \theta$
- $v = \ell\,\dot{\theta}$
- $T = \frac{1}{2} m\ell^2\dot{\theta}^2$
- $U = -mg\ell\cos\theta$
- $L = T - U = \frac{1}{2} m\ell^2\dot{\theta}^2 + mg\ell\cos\theta$
- $\frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta} \;\to\; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\ell^2\ddot{\theta}$
- $\frac{\partial L}{\partial \theta} = -mg\ell\sin\theta$
- E-L $\to m\ell^2\ddot{\theta} + mg\ell\sin\theta = 0$

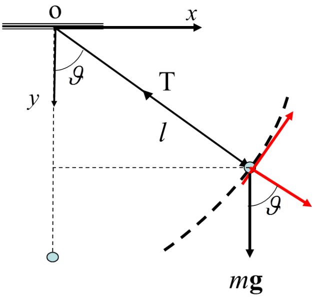
  $\to \ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$



This is great, but note that the method does not get the tension in the string since ℓ is a constraint (see next slide).

26.3.1 Dealing with forces of constraint

For the simple pendulum using Euler-Lagrange equation. The method did not get the tension in the string since ℓ was constrained. If we need to find the string tension, we need to include the radial term into the Lagrangian and to include a potential function to represent the tension:



- $\ell \to r$, add $\frac{1}{2}m\dot{r}^2$, add $V(r)$
- $L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta - V(r)$
- $\frac{\partial L}{\partial \dot{r}} = m\dot{r} \;\to\; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$
- $\frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mg\cos\theta - \frac{\partial V(r)}{\partial r}$
- $-\frac{\partial V(r)}{\partial r} = (-T)$ with T in the $-\hat{\mathbf{r}}$ dir$^{\text{n}}$.
- E-L $\to m\ddot{r} = mr\dot{\theta}^2 + mg\cos\theta - T$
- Reintroduce $\ddot{r} = 0$ and $r = \ell$; $v = r\dot{\theta}$

$$\underbrace{\frac{mv^2}{r}}_{\text{Centripetal force}} = \underbrace{T}_{\text{Tension}} - \underbrace{mg\cos\theta}_{\text{Weight}} \quad \text{as expected from NII}$$

26.3.2 The Lagrange multiplier method

An alternative method of dealing with constraints.
Back to the simple pendulum using Euler-Lagrange equation $\cdots$
Before : single variable $q_k \to \theta$. This time take TWO variables x, y but introduce a constraint into the equation. $L = T - U$

- $L' = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)+mgy+\frac{1}{2}\lambda(x^2+y^2-\ell^2)$

  λ is the *Lagrange multiplier*

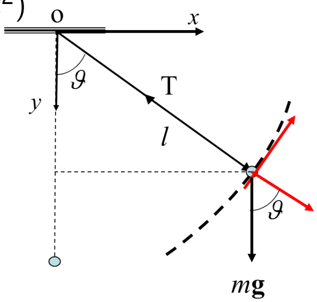
- $\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) = \frac{\partial L'}{\partial q_i}$ (including λ)

  x coord. $\to$ $m\ddot{x} = \lambda x$ (1)

  y coord. $\to$ $m\ddot{y} = mg + \lambda y$ (2)

  λ coord. $\to$ $x^2 + y^2 - \ell^2 = 0$ (3)

  (which reproduces the constraint)



Comparing with Newton II : $m\ddot{x} = -\frac{Tx}{\ell}$; $m\ddot{y} = mg - \frac{Ty}{\ell}$.
We see from the NII approach the Lagrange multiplier λ is proportional to the string tension $\lambda = -\frac{T}{\ell}$ and introduces force

27.1 The Lagrangian in various coordinate systems

- Cartesian coordinates

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$$

Already shown that E-L gives

$$m\ddot{x} = -\frac{\partial U}{\partial x} \; ; \; m\ddot{y} = -\frac{\partial U}{\partial y} ; \; m\ddot{z} = -\frac{\partial U}{\partial z}$$

$$\rightarrow \quad m\,\underline{\ddot{\mathbf{r}}} = -\nabla U$$

- Cylindrical coordinates

$$x = r\cos\phi \; ; \; \dot{x} = \dot{r}\cos\phi - r\sin\phi\,\dot{\phi}$$

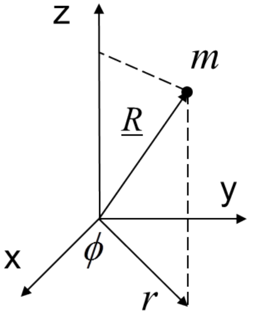
$$y = r\sin\phi \; ; \; \dot{y} = \dot{r}\sin\phi + r\cos\phi\,\dot{\phi}$$

$$z = z \; ; \; \dot{z} = \dot{z}$$

$$T = \frac{1}{2}m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$= \frac{1}{2}m[(\dot{r}^2\cos^2\phi + r^2\sin^2\phi\dot{\phi}^2 - 2r\dot{r}\cos\phi\sin\phi\dot{\phi})]$$
$$+(\dot{r}^2\sin^2\phi + r^2\cos^2\phi\dot{\phi}^2 + 2r\dot{r}\cos\phi\sin\phi\dot{\phi}) + \dot{z}^2]$$

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\,\dot{\phi}^2 + \dot{z}^2) - U(r, \phi, z)$$



Cylindrical coords

ϕ is cyclic if $U = U(r)$ only

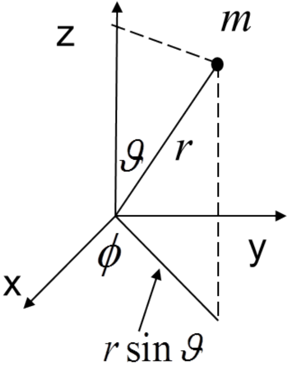
$$\rightarrow \quad p_\phi = \frac{\partial L}{\partial\dot{\phi}} = mr^2\dot{\phi}$$

$\rightarrow$ constant angular momentum

The Lagrangian in various coordinate systems continued

- Spherical coordinates

$$x = r\sin\theta\cos\phi\ ;\ \dot{x} = \dot{r}\sin\theta\cos\phi + r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi}$$
$$y = r\sin\theta\sin\phi\ ;\ \dot{y} = \dot{r}\sin\theta\sin\phi + r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\phi}$$
$$z = r\cos\theta\ ;\ \dot{z} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$



Spherical coords

$$T = \tfrac{1}{2}m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$T = \tfrac{1}{2}m\,(\dot{r}^2 + r^2\,\dot{\theta}^2 + (r\sin\theta)^2\,\dot{\phi}^2)$$

+ cross terms which all sum to zero

$$L = \tfrac{1}{2}m(\dot{r}^2 + r^2\,\dot{\theta}^2 + (r\sin\theta)^2\,\dot{\phi}^2) - U(r,\theta,\phi)$$

27.2 Example 1: the rotating bead

A bead of mass m slides on a frictionless wire which rotates about a vertical axis at an angular velocity ω. The wire is tilted away from the vertical by an angle α. Describe the motion of the bead.

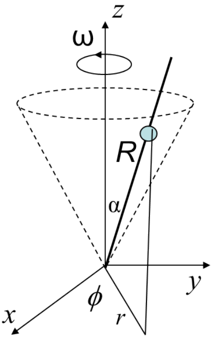
- Use spherical coordinates
- From before :

  $$T = \frac{1}{2}m\left(\dot{R}^2 + R^2\dot{\alpha}^2 + (R\sin\alpha)^2\dot{\phi}^2\right)$$

- But $\dot{\alpha} = 0$, $\dot{\phi} = \omega =$ constant
- $T = \frac{1}{2}m\left(\dot{R}^2 + R^2\omega^2\sin^2\alpha\right)$

  $U = mgR\cos\alpha$ (Take $U = 0$ at $R = 0$)
- $L = T - U$

  $$L = \frac{1}{2}m\left(\dot{R}^2 + R^2\omega^2\sin^2\alpha\right) - mgR\cos\alpha$$



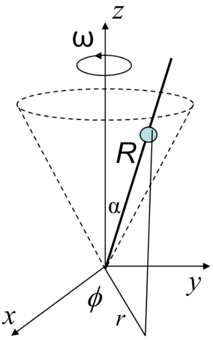
The rotating bead, continued

- $L = \frac{1}{2} m \left(\dot{R}^2 + R^2 \omega^2 \sin^2 \alpha \right) - mgR \cos \alpha$
- Single generalized coordinate R
- E-L equation:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{R}} \right) = \frac{\partial L}{\partial R}$$

$$\frac{\partial L}{\partial R} = mR\omega^2 \sin^2 \alpha - mg \cos \alpha$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{R}} \right) = \frac{d}{dt} \left(m\dot{R} \right) = m\ddot{R}$$



  - $\ddot{R} - R\omega^2 \sin^2 \alpha = -g \cos \alpha$
  - Solution : $R = Ae^{-\lambda t} + Be^{+\lambda t} + R_0$ where $\lambda = \omega \sin \alpha$

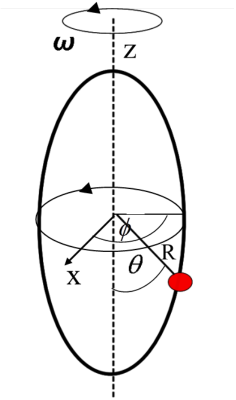
    [$P.I. \rightarrow \ \ddot{R} = 0 \rightarrow R_0 = \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha}$]

If $\dot{R} = 0$ at $t = 0$, $A = B$; then if $R = R_1$ at $t = 0$, $A = B = \frac{1}{2}(R_1 - R_0)$

  - If $\ddot{R} = 0 \rightarrow R = R_0 = \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha} \ \rightarrow$ circular motion

27.3 Example 2: bead on rotating hoop

A vertical circular hoop of radius R rotates about a vertical axis at a constant angular velocity ω. A bead of mass m can slide on the hoop without friction. Describe the motion of the bead.



- Use spherical coordinates again
- From before :

  $$T = \frac{1}{2}m\left(\dot{R}^2 + R^2\dot{\theta}^2 + (R\sin\theta)^2\dot{\phi}^2\right)$$

- But $\dot{R} = 0$, $\dot{\phi} = \omega = \text{constant}$

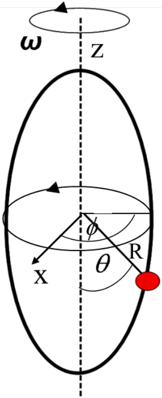
  $$T = \frac{1}{2}m(R^2\,\dot{\theta}^2 + (R\sin\theta)^2\,\omega^2) \quad (\text{NB. } \dot{R} = 0)$$

- $U = -mgR\cos\theta$ $\;(U = 0 \text{ at } \theta = 90^\circ)$
- $L = T - U$

  $$L = \frac{1}{2}m(R^2\,\dot{\theta}^2 + (R\sin\theta)^2\,\omega^2) + mgR\cos\theta$$

  One single generalized coordinate : θ

Bead on rotating hoop, continued



$L = \frac{1}{2}m(R^2\,\dot{\theta}^2 + R^2\sin^2\theta\,\omega^2) + mgR\cos\theta$

- E-L equation: $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{\partial L}{\partial\theta}$
- $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{d}{dt}\left(m\,R^2\dot{\theta}\right) = m\,R^2\ddot{\theta}$

$\frac{\partial L}{\partial\theta} = m\,R^2 sin\,\theta\cos\theta\,\omega^2 - mgR\sin\theta$

$\rightarrow\;\; \ddot{\theta} = sin\,\theta\cos\theta\,\omega^2 - \frac{g}{R}\sin\theta$

$\rightarrow\;\; \ddot{\theta} + \left(\omega_0^2 - \omega^2\,\cos\theta\right)\sin\theta = 0$

where $\omega_0^2 = \frac{g}{R}$

- If $\omega = 0$, $\ddot{\theta} + \omega_0^2\,\sin\theta = 0\;\; \rightarrow$ SHM, back to pendulum formula
- If $\omega \neq 0$, for equilibrium, a necessary condition : $\ddot{\theta} = 0$

$\rightarrow\;\; \theta = 0$ (stable equilibrium provided $\omega^2 R \leq g$),

$\rightarrow\;\; \theta = \pi$ (unstable equilibrium)

$\rightarrow\;\; \cos\theta = \frac{\omega_0^2}{\omega^2} = \frac{g}{\omega^2 R}$
(stable equilibrium about a circle provided $\omega^2 R \geq g$)

28.1 Hamilton mechanics

- Lagrangian mechanics : Allows us to find the equations of motion for a system in terms of an arbitrary set of generalized coordinates
- Now extend the method due to Hamilton
  $\rightarrow$ use of the conjugate (generalized) momenta
  $p_1, p_2, \cdots, p_n$ replace the generalized velocities
  $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$
- This has advantages when some of conjugate momenta are constants of the motion and it is well suited to finding conserved quantities
- From before, conjugate momentum : $p_k = \frac{\partial L}{\partial \dot{q}_k}$
  and E-L equation reads for coordinate k : $\dot{p}_k = \frac{\partial L}{\partial q_k}$
  (since E-L is $\dot{p}_k = \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_k}) = \frac{\partial L}{\partial q_k}$)

The Hamiltonian, continued

- Lagrangian $L = L(q_k, \dot{q}_k, t) \quad \Longrightarrow$
- $\frac{dL}{dt} = \frac{\partial L}{\partial t} + \sum_k \left(\frac{\partial L}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial L}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt} \right)$
  $= \frac{\partial L}{\partial t} + \sum_k (\frac{\partial L}{\partial q_k}\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k)$

- An aside: use rules of partial differentiation:
- If $f = f(x, y, z)$
- $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial z}\frac{dz}{dx}$

  - Conjugate momentum definition : $p_k = \frac{\partial L}{\partial \dot{q}_k}$, $\dot{p}_k = \frac{\partial L}{\partial q_k}$
  - Therefore $\frac{dL}{dt} = \frac{\partial L}{\partial t} + \sum_k (\underbrace{\dot{p}_k \dot{q}_k + p_k \ddot{q}_k}_{\frac{d}{dt}(p_k \dot{q}_k)})$
  - $\frac{d}{dt}(\underbrace{L - \sum_k p_k \dot{q}_k}_{-\,H}) = \frac{\partial L}{\partial t}$ $\quad \boxed{\frac{dH}{dt} = -\frac{\partial L}{\partial t}}$
  - Define Hamiltonian $\boxed{H = \sum_k p_k \dot{q}_k - L}$
  - If L does not depend *explicitly* on time, H is a constant of motion

28.2 The physical significance of the Hamiltonian

- From before : $H = \sum_k p_k \dot{q}_k - L$
- Where conjugate momentum : $p_k = \frac{\partial L}{\partial \dot{q}_k}$, $\dot{p}_k = \frac{\partial L}{\partial q_k}$
- Take kinetic energy $T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$
- $L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) - U(x, y, z)$
- $H = \sum_k p_k \dot{q}_k - L = \frac{1}{2} m (2\dot{x}.\dot{x} + 2\dot{y}.\dot{y} + 2\dot{z}.\dot{z}) - (T - U)$

  $= 2T - (T - U) = T + U = E \quad \rightarrow$ total energy
- From before $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$

  $\rightarrow$ If L does not depend *explicitly* on time $\frac{dH}{dt} = 0$

  $\rightarrow$ energy is a constant of the motion
- Can show by differentiation :

  Hamilton Equations $\rightarrow$ $\boxed{\frac{\partial H}{\partial p_k} = \dot{q}_k \; ; \; \frac{\partial H}{\partial q_k} = -\dot{p}_k}$

  If a coordinate does not appear in the Hamiltonian it is *cyclic* or *ignorable*

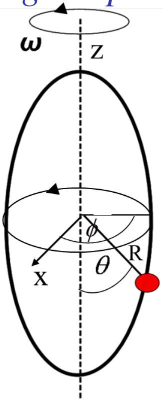
28.3 Example: re-visit bead on rotating hoop

First take the case of a free (undriven) system

- $L = \frac{1}{2} m \left(R^2 \dot{\theta}^2 + R^2 \sin^2 \theta \, \dot{\phi}^2 \right) + mgR \cos \theta$
- $H = \sum_k p_k \dot{q}_k - L \;\; ; \;\; p_k = \frac{\partial L}{\partial \dot{q}_k}$
- $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m R^2 \dot{\theta} \;\; ; \; p_\phi = mR^2 \sin^2 \theta \, \dot{\phi}$
- $H = m R^2 \dot{\theta}^2 + mR^2 \sin^2 \theta \, \dot{\phi}^2 - L$

  $= \frac{1}{2} m \left(R^2 \dot{\theta}^2 + R^2 \sin^2 \theta \, \dot{\phi}^2 \right) - mgR \cos \theta$

  $\rightarrow \;\; H = T + U = E$



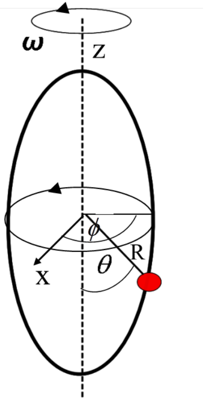
L does not depend explicitly on t,
H, E conserved $\rightarrow$ Hamiltonian gives the total energy

Hamilton Equations : $\dot{q}_k = \frac{\partial H}{\partial p_k} \;\; ; \;\; \dot{p}_k = -\frac{\partial H}{\partial q_k}$

$\rightarrow \;\; \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$ (ignorable)

$\rightarrow \;\; p_\phi = mR^2 \sin^2 \theta \, \dot{\phi} = J_z$ = constant of the motion

Example continued



Now consider a DRIVEN system - hoop rotating at constant angular speed ω

- $L = \frac{1}{2} m\,(R^2\,\dot{\theta}^2 + R^2\omega^2 \sin^2\theta) + mgR\cos\theta$
- $H = \sum_k p_k \dot{q}_k - L \;\; ; \;\; p_k = \frac{\partial L}{\partial \dot{q}_k}$
- $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\,R^2\dot{\theta}$; a single coordinate θ
- $H = m\,R^2\dot{\theta}^2 - L$

  $= \frac{1}{2} m \left(R^2\dot{\theta}^2 - R^2\omega^2 \sin^2\theta \right) - mgR\cos\theta$
- $E = \frac{1}{2} m\,(R^2\,\dot{\theta}^2 + R^2\omega^2 \sin^2\theta) - mgR\cos\theta$

  Hence $E = H + mR^2\omega^2 \sin^2\theta$

  $\rightarrow \;\; E\,(= T + U) \neq H$

- $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$
- H is a constant of the motion, E is not const.

So what's different ?

In this case the hoop has been forced to rotate at an angular velocity ω. External energy is being supplied to the system.

28.4 Noether's theorem

The theorem states : *Whenever there is a continuous symmetry of the Lagrangian, there is an associated conservation law.*

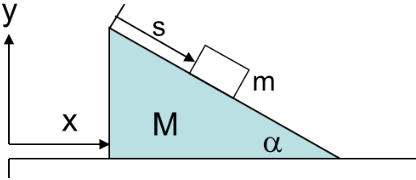
- *Symmetry* means a transformation of the generalized coordinates q_k and $\dot{q}_k$ that leaves the value of the Lagrangian unchanged.
- If a Lagrangian does not depend on a coordinate q_k (ie. is cyclic) it is invariant (*symmetric*) under changes $q_k \to q_k + \delta q_k$; the corresponding generalized momentum $p_k = \frac{\partial L}{\partial \dot{q}_k}$ is conserved

1. For a Lagrangian that is symmetric under changes $t \to t + \delta t$, the total energy H is conserved $\to H = \sum_k \frac{\partial L}{\partial \dot{\mathbf{q}}} \dot{\mathbf{q}} - L$
2. For a Lagrangian that is symmetric under changes $r \to r + \delta r$, the linear momentum $\underline{\mathbf{p}}$ is conserved
3. For a Lagrangian that is symmetric under small rotations of angle $\theta \to \theta + \delta\theta$ about an axis $\underline{\hat{\mathbf{n}}}$ such a rotation transforms the Cartesian coordinates by $\underline{\mathbf{r}} \to \underline{\mathbf{r}} + \delta\theta\, \underline{\hat{\mathbf{n}}} \times \underline{\mathbf{r}}$, the conserved quantity is the component of the angular momentum $\underline{\mathbf{J}}$ along the $\underline{\hat{\mathbf{n}}}$ axis

29.1 Re-examine the sliding blocks using E-L

A block of mass m slides on a frictionless inclined plane of mass M, which itself rests on a horizontal frictionless surface. Find the acceleration of the inclined plane.

- Reduce the problem to two generalized coordinates, x and s
- Motion of the inclined plane :
  $T_M = \frac{1}{2}M\dot{x}^2$
- Motion of the block :
  $T_m = \frac{1}{2}m(\dot{x}'^2 + \dot{y}'^2)$ where
- $x' = x + s\cos\alpha$; $y' = -s\sin\alpha$
- $\dot{x}' = \dot{x} + \dot{s}\cos\alpha$; $\dot{y}' = -\dot{s}\sin\alpha$
  - $T_m = \frac{1}{2}m\left[(\dot{x} + \dot{s}\cos\alpha)^2 + \frac{1}{2}m(\dot{s}\sin\alpha)^2\right]$
  - $T = T_m + T_M = \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}m\left(\dot{s}^2 + 2\dot{x}\dot{s}\cos\alpha\right)$
  - $U = -m\,g\,s\sin\,\alpha$



Sliding blocks, continued

- Lagrangian $\quad L = T - U$
- $L = \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}m\left(\dot{s}^2 + 2\dot{x}\dot{s}\cos\alpha\right) + mgs\sin\alpha$
- 2 generalized coordinates $\rightarrow$ x and s
- The E-L equation $\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$
- E-L for x : $\frac{\partial L}{\partial x} = 0$; $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}\left[(m+M)\dot{x} + m\dot{s}\cos\alpha\right]$

  $\rightarrow \quad (m+M)\ddot{x} + m\ddot{s}\cos\alpha = 0 \qquad\qquad (1)$
- E-L for s : $\frac{\partial L}{\partial s} = mg\sin\alpha$; $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = \frac{d}{dt}\left[m\left(\dot{s} + \dot{x}\cos\alpha\right)\right]$

  $\rightarrow \quad \ddot{s} + \ddot{x}\cos\alpha = g\sin\alpha \qquad\qquad (2)$
- Rearranging (1) & (2)

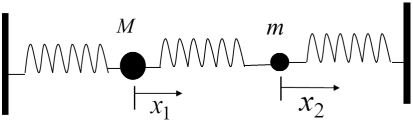
$$\ddot{x} = -g\,\frac{\sin\alpha\cos\alpha}{\sin^2\alpha + M/m} \quad ; \quad \ddot{s} = g\,\frac{\sin\alpha(1+M/m)}{\sin^2\alpha + M/m}$$

- From (1) $\qquad M\dot{x} + m(\dot{x} + \dot{s}\cos\alpha) = \text{const.}$

  $\rightarrow \quad M\dot{x} + m\dot{x}' = \text{const.}$ Conservation of momentum.

29.2 Normal modes of coupled identical springs

Coupled identical springs mounted horizontally. x_1 and x_2 measure displacements from the respective equilibrium positions. Assume the springs are unstretched at equilibrium.



- The problem has two generalized coordinates, x_1 and x_2

  $T = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$

  $U = \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}k\left(x_2 - x_1\right)^2$
- $L = T - U$
  - E-L equation for x_1 : $\frac{\partial L}{\partial x_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = 0$
  - $\frac{\partial L}{\partial x_1} = -kx_1 + k(x_2 - x_1) = k(x_2 - 2x_1)$; $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = M\ddot{x}_1$

    $\rightarrow \quad M\ddot{x}_1 = k(x_2 - 2x_1)$; $m\ddot{x}_2 = k(x_1 - 2x_2)$
  - $\begin{pmatrix} M\ddot{x}_1 \\ m\ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Coupled identical springs, continued

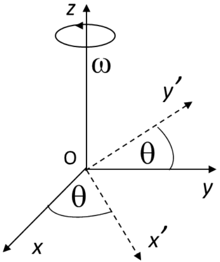
- $$\begin{pmatrix} M\,\ddot{x}_1 \\ m\,\ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad (1)$$
- SHM solutions $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} exp(\,i\,\omega\,t)$
- Substitute into (1)

$$-\omega^2 \underbrace{\begin{pmatrix} M & 0 \\ 0 & m \end{pmatrix}}_{\underline{\mathbf{M}}} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = -\underbrace{\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}}_{\underline{\mathbf{K}}} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

- Putting $\omega^2 = \lambda \;\; \rightarrow \qquad \underline{\mathbf{M}}^{-1}\underline{\mathbf{K}}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \lambda\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$
- Eigenvalue equation; homogeneous solutions
- Etc etc ....

29.3 Final example: a rotating coordinate system

- Lagrangian of a free particle :
  $L = \frac{1}{2} m \dot{\underline{\mathbf{r}}}^2$, $\underline{\mathbf{r}} = (x, y, z)$ (with $U = 0$)
- Measure the motion w.r.t. a coordinate system rotating with angular velocity $\underline{\omega} = (0, 0, \omega)$ about the z axis.
- $\underline{\mathbf{r}}' = (x', y', z')$ are coordinates in the rotating system



- $$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
- Take the inverse :
  $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$
- Substitute these expressions into the Lagrangian above $\rightarrow$ find L in terms of the rotating coordinates

A rotating coordinate system, continued

- $L = \frac{1}{2} m \left[(\dot{x}' - \omega\, y')^2 + (\dot{y}' + \omega\, x')^2 + \dot{z}'^2 \right]$

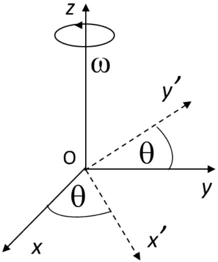
  $= \frac{1}{2} m (\dot{\underline{\mathbf{r}}}' + \underline{\omega} \times \underline{\mathbf{r}}')^2$ in the general case
- In this rotating frame, we can use Lagrange equations to derive the equations of motion. Taking derivatives, we have
- $\frac{\partial L}{\partial \underline{\mathbf{r}}'} = m \left[\dot{\underline{\mathbf{r}}}' \times \underline{\omega} - \underline{\omega} \times (\underline{\omega} \times \underline{\mathbf{r}}') \right]$

  where $\frac{\partial}{\partial \underline{\mathbf{r}}'} = \left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'} \right)$

  $$\rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{\mathbf{r}}}'} \right) = m \frac{d}{dt} (\dot{\underline{\mathbf{r}}}' + \underline{\omega} \times \underline{\mathbf{r}}') = m (\ddot{\underline{\mathbf{r}}}' + \underline{\omega} \times \dot{\underline{\mathbf{r}}}')$$

  - So the Lagrange equation becomes

  $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{\mathbf{r}}}'} \right) - \frac{\partial L}{\partial \underline{\mathbf{r}}'} = m \Big[\underbrace{\ddot{\underline{\mathbf{r}}}'}_{\text{radial force}} + \underbrace{\underline{\omega} \times (\underline{\omega} \times \underline{\mathbf{r}}')}_{\text{Centrifugal force}} + \underbrace{2\, \underline{\omega} \times \dot{\underline{\mathbf{r}}}'}_{\text{Coriolis force}} \Big] = 0$$



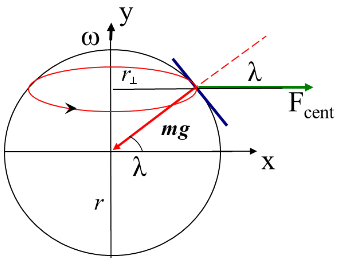
A rotating coordinate system, continued

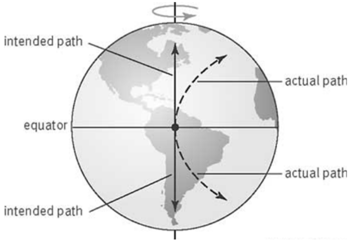
- Centrifugal and Coriolis forces are examples of “fictitious forces” :

  $\rightarrow$ called “fictitious” since they are a consequence of the reference frame, rather than any interaction. The forces do not exist in an inertial frame.

- The centrifugal force $\underline{\mathbf{F}}_{cent} = m\underline{\omega} \times (\underline{\omega} \times \underline{\mathbf{r}}')$ points outwards in the plane perpendicular to $\underline{\omega}$ with magnitude $m\omega^2|r'_\perp|$ ($\perp$ is the projection perpendicular to $\underline{\omega}$)

- The Coriolis force $\underline{\mathbf{F}}_{cor} = 2m\underline{\omega} \times \underline{\dot{\mathbf{r}}}'$ acts in a direction perpendicular to the rotation axis $\underline{\omega}$ and to the velocity of the body in the rotating frame



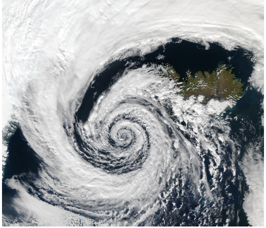


A rotating coordinate system, continued

- Coriolis force responsible for the circulation of oceans and the atmosphere.
- A projectile thrown in the northern hemisphere rotates in a clockwise direction
- A projectile thrown in the southern hemisphere rotates in an anti-clockwise direction.
- For a particle moving along the equator, $\underline{\omega} \perp \underline{\dot{\mathbf{r}}}'$, the Coriolis force tends to zero $\rightarrow$ no effect on the projectile
- The Coriolis force is responsible for the formation of hurricanes. These rotate in different directions in the northern and southern hemisphere. They never form within 500 miles of the equator where the Coriolis force is too weak.

Australia

Iceland



THE END